Final Exam
Math 317
December 14th, 2017

Last Name: Solutions  First Name:  

Student #:  Instructor’s Name:  

Instructions:
No memory aids allowed. No calculators allowed. No communication devices allowed. Use the space provided on the exam. If you use the back of a page, write “see back” on the front of the page. This exam is 180 minutes long.

Rules governing examinations
- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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1. Let \( \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}, \quad r = |\vec{r}|. \)

(a) **3 points** Compute \( a \) where \( \text{grad} \left( \frac{1}{r} \right) = -r^a \vec{r}. \)

\[
\frac{\partial}{\partial x} \left( \frac{1}{\sqrt{x^2+y^2+z^2}} \right) = -\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{x}{r^3}
\]

Similarly \( \frac{\partial}{\partial y} \left( \frac{1}{r} \right) = -\frac{y}{r^3} \)

\( \frac{\partial}{\partial z} \left( \frac{1}{r} \right) = -\frac{z}{r^3} \)

So \( \nabla \left( \frac{1}{r} \right) = -\frac{1}{r^3} \langle x, y, z \rangle = -r^{-3} \vec{r} \)

**Answer:**
\[
a = -3
\]

(b) **3 points** Compute \( a \) where \( \text{div}(r \vec{r}) = a r \)

\[
\text{div}(r \vec{r}) = (\nabla r) \cdot \vec{r} + r \text{div} \vec{r}
\]

\[
= \frac{\vec{r} \cdot \vec{r}}{r} + 3r = \frac{r^2}{r} + 3r = 4r
\]

\( \vec{r} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle \)

\[
\nabla r = \frac{\vec{r}}{r}
\]

\( \text{div} \left( \langle x, y, z \rangle \right) = 3 \)

**Answer:**
\[
a = 4
\]

(c) **3 points** Compute \( a \) where \( \text{div} (\text{grad} (r^3)) = a \, r. \)

\[
\frac{\partial}{\partial x} r^3 = 3r^2 \frac{\partial r}{\partial x} = 3r^2 \frac{x}{r} = 3rx
\]

\( \nabla r^3 = 3r \vec{r} \)

\( \text{div} (3r \vec{r}) = 3 \, \text{div} (r \vec{r}) = 3 \cdot 4r = 12r \)

**Answer:**
\[
a = 12
\]
2. **5 points** Let \( \mathbf{r}(t) = \left( \frac{1}{3}t^3, \frac{1}{2}t^2, \frac{1}{2}t \right) \quad t \geq 0 \). Compute \( s(t) \), the arclength of the curve at time \( t \).

\[
\mathbf{r}'(t) = \left< t^2, t, \frac{1}{2} \right>
\]

\[
|\mathbf{r}'(t)| = \sqrt{t^4 + t^2 + \frac{1}{4}} = \sqrt{(t^2 + \frac{1}{2})^2} = t^2 + \frac{1}{2}
\]

\[
s(t) = \int_0^t \left( u^2 + \frac{1}{2} \right) \, du
\]

\[
= \left[ \frac{u^3}{3} + \frac{u}{2} \right]_0^t = \frac{t^3}{3} + \frac{t}{2}
\]

\[
s(t) = \frac{t^3}{3} + \frac{t}{2}
\]
3. A particle of mass \( m = 2 \)

is acted on by a force

\[
\vec{F} = \langle 4t, 6t^2, -4t \rangle.
\]

The particle has velocity zero at \( t = 0 \) and is located at the point \((1, 2, 3)\).

(a) [2 points] Find the velocity vector \( \vec{v}(t) \) for \( t \geq 0 \).

\[
\vec{F} = m \vec{a} \quad \vec{a} = \vec{v}'(t) = \frac{\vec{F}}{m} = \langle 2t, 3t^2, -2t \rangle
\]

\[
\Rightarrow \vec{v}(t) = \langle t^2, t^3, -t^2 \rangle + \vec{c} \quad \text{since} \quad \vec{v}(0) = \vec{0}
\]

\[
\vec{v}(t) = \langle t^2, t^3, -t^2 \rangle
\]

(b) [2 points] Find the position vector \( \vec{r}(t) \) for \( t \geq 0 \).

\[
\vec{v} = \vec{r}'(t) = \langle t^2, t^3, -t^2 \rangle
\]

\[
\Rightarrow \vec{r}(t) = \langle \frac{1}{3} t^3, \frac{1}{4} t^4, -\frac{1}{3} t^3 \rangle + \vec{c}
\]

\[
\text{since} \quad \vec{r}(0) = \langle 1, 2, 3 \rangle \quad \vec{c} = \langle 1, 2, 3 \rangle
\]

\[
\vec{r}(t) = \langle \frac{1}{3} t^3 + 1, \frac{1}{4} t^4 + 2, -\frac{1}{3} t^3 + 3 \rangle
\]
(c) **4 points** Find $\kappa(t)$ the curvature of the path traversed by the particle for $t \geq 0$.

\[
|\vec{\tau}(t)| = |\vec{\tau}'(t)| = |t^2 < 1, t, -1> | = t^2 \sqrt{1 + t^2}
\]

\[
|\vec{\tau}' \times \vec{\tau}''| = \sqrt{\frac{t^2}{t^2 + t^2}} = \sqrt{\frac{t^2}{2t^0}} = t \sqrt{2 - 1}
\]

\[
|\frac{\vec{\tau}' \times \vec{\tau}''}{|\vec{\tau}''|}| = \frac{|t^4, 0, t^9|}{t^6 (2 + t^2)^{3/2}} = \frac{\sqrt[3]{t^2}}{t^6 (2 + t^2)^{3/2}}
\]

(d) **3 points** Find the work done by the force on the particle from $t = 0$ to $t = T$.

\[
W = \int_0^T \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_0^T <4t, 6t^2, -4t> \cdot <t^2, t^3, -t^2> dt
\]

\[
= \int_0^T (4t^3 + 6t^5 + 4t^3) dt = \left[ t^4 + t^6 + t^4 \right]_0^T
\]

\[
= 2T^4 + T^6
\]
4. Let

\[ \mathbf{F} = \left( \frac{2z}{1+y} + \sin(x^2), \frac{3z}{1+x} + \sin(y^2), 5(x+1)(y+2) \right) \]

Let \( C \) be the oriented curve consisting of the four line segments from \((0,0,0)\) to \((2,0,0)\), from \((2,0,0)\) to \((0,0,2)\), from \((0,0,2)\) to \((0,3,0)\), and from \((0,3,0)\) to \((0,0,0)\).

(a) **3 points** Draw a picture of \( C \). Clearly indicate the orientations on each line segment.

(b) **6 points** Compute the work integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

Use Stoke's theorem.

Let \( S = S_1 + S_2 \) where \( S_1 \) and \( S_2 \) are the triangles indicated in the picture with the orientations \( \mathbf{N}_1 = -\hat{z} \quad \mathbf{N}_2 = -\hat{z} \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl} \mathbf{F} \cdot (-\hat{z}) \, dS + \iint_{S_2} \text{curl} \mathbf{F} \cdot (-\hat{z}) \, dS
\]

\[
= \iint_{S_1} \left( \frac{3}{1+x} - 5(x+1) \right) \, dS + \iint_{S_2} \left( 5(y+2) - \frac{2}{1+y} \right) \, dS
\]

\( x = 0 \) on \( S_1 \), \( y = 0 \) on \( S_2 \), so

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} (3-5) \, dS + \iint_{S_2} (10-2) \, dS = -2 \text{Area}(S_1) + 8 \text{Area}(S_2)
\]

\[
= -2 \cdot 3 + 8 \cdot 2 = 10
\]
5. **5 points** Recall that if \( \mathbf{T} \) is the unit tangent vector to an oriented curve with arclength parameter \( s \), then the curvature \( \kappa \) and the principal normal vector \( \mathbf{N} \) are defined by the equation

\[
\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}
\]

Moreover, the torsion \( \tau \) and the binormal vector \( \mathbf{B} \) are defined by the equations

\[
\mathbf{B} = \mathbf{T} \times \mathbf{N}, \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}.
\]

Show that

\[
\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}.
\]

So \( \mathbf{N} = \mathbf{B} \times \mathbf{T} \) or \( \mathbf{B} \times \mathbf{N} = -\mathbf{T} \)

\[
\Rightarrow \quad \frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds}
\]

\[
= -\tau \mathbf{N} \times \mathbf{T} + \mathbf{B} \times (\kappa \mathbf{N})
\]

\[
= \tau \mathbf{T} \times \mathbf{N} + \kappa \mathbf{B} \times \mathbf{N}
\]

\[
= \tau \mathbf{B} - \kappa \mathbf{T}
\]

\[
\therefore \quad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}
\]
6. Let $a, b, c$ be constants and let

$$
\vec{F} = \langle 2bx^2y^2 - axz, 2ax^3y - 3cz^2, 2byz + x^2 \rangle.
$$

(a) [2 points] Compute curl $\vec{F}$.

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2b x^2 y^2 - axz & 2ax^3y - 3cz^2 & 2byz + x^2 \\
\end{vmatrix}
\]

\[= \langle 2b + 6cz, -ax - 2x, 6ax^2y - 4bz^2 \rangle
\]

(b) [3 points] Find the values of $a, b, c$ which make $\vec{F}$ conservative.

Since domain of $\vec{F}$ is all of $\mathbb{R}^3$ (so simply connected)

$\vec{F}$ is cons $\iff$ curl $\vec{F} = \vec{0}$

\[\Rightarrow (2b+6c)z = 0 \quad (-a-2)x = 0 \quad (6a-4b)x^2 y = 0\]

\[\begin{align*}
\text{(1)} \Rightarrow & \quad 2b + 6c = 0 \\
\text{(2)} \Rightarrow & \quad a = -2 \quad \text{so then} \quad \text{(3)} \Rightarrow 6(-2) - 4b = 0 \Rightarrow 4b = -12 \Rightarrow b = -3 \\
\text{(4)} \Rightarrow & \quad 2(-3) + 6c = 0 \Rightarrow 6c = 6 \Rightarrow c = 1
\end{align*}\]

\[a = -2 \quad b = -3 \quad c = 1\]
(c) **3 points** Using the values of \( a, b, c \) from the previous part, find a potential function of \( \mathbf{F} \)

\[
\nabla f = \langle -6x^2y^2 + 2xz, -4x^3y - 3z^2, -6y^2 + x^2 \rangle
\]

\[f_x = -6x^2y^2 + 2xz \implies f = -2x^3y^2 + x^2z + g(y,z)\]

\[\implies f_y = -4x^3y + g_y = -4x^3y - 3z^2 \implies g_y = -3z^2 \implies g = -3z^2y + h(z)\]

\[\implies f = -2x^3y^2 + x^2z - 3z^2y + h(z)\]

\[\implies f_z = z^2 - 6zy + h(z) = -6yz + x^2 \implies h(z) = 0\]

\[
f(x,y,z) = -2x^3y^2 + x^2z - 3z^2y
\]

---

(d) **2 points** Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(t) = (\cos(t), 2\sin(t), \frac{2t}{\pi} + 1) \) for \( 0 \leq t \leq \frac{\pi}{2} \).

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{F}(\frac{\pi}{2})) - f(\mathbf{F}(0))
\]

\[= f\left( \cos\left(\frac{\pi}{2}\right), 2\sin\left(\frac{\pi}{2}\right), 2 \right) - f\left( \cos(0), 2\sin(0), 1 \right)\]

\[= f\left( 0, 2, 2 \right) - f\left( 1, 0, 1 \right)\]

\[= -3 \cdot 2^2 \cdot 2 - 1 = -24 - 1 = -25\]
7. (a) 4 points Evaluate

\[ \int_C \sqrt{1 + x^2} \, dx + (2xy^2 + y^2) \, dy \]

where \( C \) is the unit circle \( x^2 + y^2 = 1 \) oriented counterclockwise.

\[ P = \sqrt{1 + x^3} \quad Q = 2xy^2 + y^2 \]

\[ Q_x - P_y = 2y^2 \]

\[ \int_C P \, dx + Q \, dy = \iint_R 2y^2 \, dx \, dy \]

\[ C \]

\[ = \int_0^{2\pi} \int_0^1 2(r^2 \sin^2 \theta) \, r \, dr \, d\theta = \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^1 2r^3 \, dr \]

\[ = \pi \left[ \frac{r^4}{2} \right]_0^1 = \frac{\pi}{2} \]

(b) 4 points Evaluate

\[ \int_C \sqrt{1 + x^3} \, dx + (2xy^2 + y^2) \, dy \]

where now \( C \) is the part of the unit circle \( x^2 + y^2 = 1 \) with \( x \geq 1 \), still oriented counterclockwise.

Let \( C' \) be the segment from \((0, -1) \) to \((0, 1) \) and \( R \) the region shown so that \( DR = C - C' \)

\[ \iint_R 2y^2 \, dx \, dy = \int_C P \, dx + Q \, dy - \int_{C'} P \, dx + Q \, dy \]

\[ \int_0^{2\pi} \int_0^1 2(r^2 \sin^2 \theta) r \, dr \, d\theta = \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^1 2r^3 \, dr \]

\[ = \pi \left[ \frac{r^4}{2} \right]_0^1 = \frac{\pi}{2} \]

So

\[ \int_C P \, dx + Q \, dy = \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^1 2r^3 \, dr + \int_{y=-1}^1 y^2 \, dy \]

\[ = \frac{\pi}{2} \cdot \frac{1}{2} + \left[ \frac{y^3}{3} \right]_{-1}^1 = \frac{\pi}{4} + \frac{2}{3} \]
8. Let $S$ be the part of the sphere $x^2 + y^2 + z^2 = 2$ where $y \geq 1$, oriented away from the origin.

(a) 4 points Compute

\[ \int \int_S y^3 \, dS \]

Parameterize with $x, z$:

\[ \mathbf{r}(r, \theta) = \langle r \cos \theta, \sqrt{2-r^2}, r \sin \theta \rangle \]

Domain in the $xz$ plane is given by

\[ y = \sqrt{2-x^2-z^2} \geq 1 \Rightarrow 2-x^2-z^2 \leq 1 \]

\[ \Rightarrow x^2+z^2 \leq 1 \]

\[ \mathbf{F}_x = \langle 1, \frac{-x}{\sqrt{2-x^2-z^2}}, 0 \rangle \]

\[ \mathbf{F}_z = \langle 0, \frac{-z}{\sqrt{2-x^2-z^2}}, 1 \rangle \]

\[ \mathbf{F}_x \times \mathbf{F}_z = \langle -x, \frac{-z}{\sqrt{2-x^2-z^2}}, -1 \rangle, \frac{-z}{\sqrt{2-x^2-z^2}^2} \]

\[ |\mathbf{F}_x \times \mathbf{F}_z| = \sqrt{1 + \frac{x^2}{2-x^2-z^2} + \frac{z^2}{2-x^2-z^2}} = \frac{\sqrt{2}}{\sqrt{2-x^2-z^2}} \]

\[ \int \int_S y^3 \, dS = \int \int_S (2-x^2-z^2)^{\frac{3}{2}} \frac{\sqrt{2}}{\sqrt{2-x^2-z^2}} \, dx \, dz = \sqrt{2} \int \int_S (2-x^2-z^2) \, dx \, dz \]

\[ = \frac{\sqrt{2}}{2} \int_0^1 \int_0^1 (2-r^2) \, r \, dr \, d\theta = 2\pi \sqrt{2} \left[ \left. \frac{1}{4} \frac{r^4}{4} \right|_0^1 \right]_{\theta=0}^{\theta=\frac{\pi}{2}} \]

\[ = 2\pi \sqrt{2} \left( \frac{3}{4} \right) = \frac{3\pi \sqrt{2}}{2} \]
Compute
\[
\iint_S (xy \hat{i} + xz \hat{j} + zy \hat{k}) \cdot d\mathbf{S}
\]
\[
\iint_S \langle xy, xz, zy \rangle \cdot d\mathbf{S} = -\iint_S \langle x \sqrt{2-x^2-z^2}, x \sqrt{2-x^2-z^2}, z \sqrt{2-x^2-z^2} \rangle \cdot d\mathbf{S}
\]
\[
\begin{align*}
\iint_{S^*} \langle x \sqrt{2-x^2-z^2}, x \sqrt{2-x^2-z^2}, z \sqrt{2-x^2-z^2} \rangle \cdot d\mathbf{S} &= \iint_{S^*} \left( \frac{-x}{\sqrt{2-x^2-z^2}}, -1, \frac{-z}{\sqrt{2-x^2-z^2}} \right) dxdz \\
&= \iint \left( -x^2 - x^2 - z^2 \right) dxdz \\
&= \iint \left( -x^2 - z^2 \right) dxdz \\
&= \iiint_{x^2+z^2 \leq 1} \left( x^2 + z^2 + xz \right) dxdz \\
&= \iiint_{x^2+z^2 \leq 1} \left( r^2 + r \cos \theta \sin \theta \right) drd\theta d\phi \\
&= \int_0^{2\pi} \int_0^1 \int_0^1 r^3 (1 + \cos \theta \sin \theta) r dr d\theta d\phi \\
&= \int_0^{2\pi} \left[ \frac{r^4}{4} \right]^{1}_0 \left[ \theta + \frac{1}{2} \sin^2 \theta \right]^{2\pi}_0 \\
&= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}
\end{align*}
\]
9. [5 points] Let $S$ be the sphere $x^2 + y^2 + z^2 = 3$ oriented inward. Compute the flux integral

$$ \int_S \vec{F} \cdot d\vec{S}, $$

where

$$ \vec{F} = \langle xy^2 + y^4 z^3, yz^2 + x^4 z, zx^2 + xy^4 \rangle. $$

E: $x^2 + y^2 + z^2 \leq 3$

**Inward normal**

\[ \text{Div thm:} \]

$$ \int_S \vec{F} \cdot d\vec{S} = -\int_E \text{div} \vec{F} \, d\text{vol} $$

$$ = -\int_E \left( y^2 + z^2 + x^2 \right) \, d\text{vol} $$

$$ = -\int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{3}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta $$

$$ = -\int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \int_0^{\sqrt{3}} \rho^4 \, d\rho $$

$$ = -2\pi \left[ -\cos \phi \right]_0^\pi \left[ \frac{1}{5} \rho^5 \right]_0^{\sqrt{3}} $$

$$ = -2\pi \left( 1 + 1 \right) \left( \frac{1}{5} \cdot 3^{5/2} \right) $$

$$ = -\frac{4\pi}{5} \cdot 3^{5/2} $$

$$ = \frac{36\pi}{5} \sqrt{3} $$