MATHEMATICS 317 April 2000 Final Exam Solutions

1) A particle moves along the curve $C$ of intersection of the surfaces $z^2 = 12y$ and $18x = yz$ in the upward direction. When the particle is at $(1, 3, 6)$ its velocity $\vec{v}$ and acceleration $\vec{a}$ are given by

$$\vec{v} = 6\vec{i} + 12\vec{j} + 12\vec{k} \quad \vec{a} = 27\vec{i} + 30\vec{j} + 6\vec{k}$$

a) Write a vector parametric equation for $C$ using $u = \frac{r}{6}$ as a parameter.
b) Find the length of $C$ from $(0,0,0)$ to $(1,3,6)$.
c) If $u = u(t)$ is the parameter value for the particle’s position at time $t$, find $\frac{du}{dt}$ when the particle is at $(1,3,6)$.
d) Find $\frac{d^2u}{dt^2}$ when the particle is at $(1,3,6)$.

Solution. a) Since $z = 6u$, $y = \frac{u^2}{12} = 3u^2$ and $x = \frac{u^3}{18} = u^3$,

$$\vec{r}(u) = u^3\vec{i} + 3u^2\vec{j} + 6u\vec{k}$$

$$\vec{r}'(u) = 3u^2\vec{i} + 6u\vec{j} + 6\vec{k}$$

$$\vec{r}''(u) = 6u\vec{i} + 6\vec{j}$$

$$\frac{du}{dt} = \sqrt{9u^4 + 36u^2 + 36} = 3(u^2 + 2)$$

b) 

$$\int_C ds = \int_0^1 \frac{du}{du} du = \int_0^1 3(u^2 + 2) du = [u^3 + 6u]_0^1 = 7$$

c) Denote by $\vec{R}(t)$ the position of the particle at time $t$. Then

$$\vec{R}(t) = \vec{r}(u(t)) \quad \Rightarrow \quad \vec{R}'(t) = \vec{r}'(u(t)) \frac{du}{dt}$$

In particular, if the particle is at $(1, 3, 6)$ at time 1, then $u(1) = 1$ and

$$6\vec{i} + 12\vec{j} + 12\vec{k} = \vec{R}'(1) = \vec{r}'(1) \frac{du}{dt}(1) = (3\vec{i} + 6\vec{j} + 6\vec{k}) \frac{du}{dt}(1) \quad \Rightarrow \quad \frac{du}{dt} = 2$$

d) By the product and chain rules,

$$\vec{R}''(t) = \vec{r}''(u(t)) \frac{du}{dt} + \vec{r}'(u(t)) \frac{d^2u}{dt^2}$$

In particular,

$$27\vec{i} + 30\vec{j} + 6\vec{k} = \vec{R}''(1) = \vec{r}''(1) \left(\frac{du}{dt}(1)\right)^2 + \vec{r}'(1) \frac{d^2u}{dt^2}(1) = (6\vec{i} + 6\vec{j})^2 + (3\vec{i} + 6\vec{j} + 6\vec{k}) \frac{d^2u}{dt^2}(1)$$

Simplifying

$$3\vec{i} + 6\vec{j} + 6\vec{k} = (3\vec{i} + 6\vec{j} + 6\vec{k}) \frac{d^2u}{dt^2}(1) \quad \Rightarrow \quad \frac{d^2u}{dt^2} = 1$$

2) The vector field $\vec{F}(x, y, z) = Ax^3y^2z\vec{i} + (z^3 + Bx^3yz)\vec{j} + (3yz^2 - x^4y^2)\vec{k}$ is conservative on $\mathbb{R}^3$.

a) Find the values of the constants $A$ and $B$.
b) Find a scalar field $\phi$ such that $\vec{F} = \nabla \phi$ on $\mathbb{R}^3$.
c) If $C$ is the curve $y = -x$, $z = x^2$ from $(0,0,0)$ to $(1,-1,1)$, evaluate $I = \int_C \vec{F} \cdot d\vec{r}$.
d) Evaluate $J = \int_C (z-4x^3y^2z)dx + (z^3 - x^4yz)dy + (3yz^2 - x^4y^2)dz$, where $C$ is the curve of part (c).
e) Let $T$ be the closed triangular path with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, oriented counterclockwise as seen from the point $(1,1,1)$. Evaluate $\int_T (z\vec{i} + \vec{F}) \cdot d\vec{r}$.
Solution. The field is conservative only if
\[
\frac{\partial F_1}{\partial y} = \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y}
\]
That is,
\[
\frac{\partial}{\partial y}(Ax^3y^2z) = \frac{\partial}{\partial x}(z^3 + Bx^4yz) \iff 2Ax^3yz = 4Bx^3yz
\]
\[
\frac{\partial}{\partial x}(Ax^3y^2z) = \frac{\partial}{\partial z}(3yz^2 - x^4y^2) \iff Ax^3y^2 = -4x^3y^2
\]
\[
\frac{\partial}{\partial z}(z^3 + Bx^4yz) = \frac{\partial}{\partial y}(3yz^2 - x^4y^2) \iff 3z^2 + Bx^4y = 3z^2 - 2x^4y
\]
Hence only \(A = -4, B = -2\) works.

b) When \(A = -4, B = -2\)
\[
\vec{F} = -4x^3y^2z \hat{i} + (z^3 - 2x^4yz) \hat{j} + (3yz^2 - x^4y^2) \hat{k} = \vec{\nabla}(-x^4y^2z + yz^3 + C)
\]
So \(\phi(x, y, z) = -x^4y^2z + yz^3\) is one allowed scalar potential.

c) \(I = \phi(1, -1, 1) - \phi(0, 0, 0) = -2\).

d) \(J = \int_C (z - 4x^3y^2z) dx + (z^3 - x^4yz) dy + (3yz^2 - x^4y^2) dz = \int_C (\hat{z} + x^4yz \hat{j} + \vec{F}) \cdot d\vec{r} = -2 + \int_C (\hat{z} + x^4yz \hat{j}) \cdot d\vec{r}\)

Parametrize \(C\) by \(\vec{r}(x) = x \hat{i} - x \hat{j} + x^2 \hat{k}\). As \(\frac{d\vec{r}}{dx} = \hat{i} - \hat{j} + 2x \hat{k}\)
\[
\int_C (\hat{z} + x^4yz \hat{j}) \cdot d\vec{r} = \int_0^1 (x^2 \hat{i} - x^7 \hat{j}) \cdot (\hat{i} - \hat{j} + 2x \hat{k}) \, dx = \int_0^1 (x^2 - x^7) \, dx = \frac{1}{3} - \frac{1}{8} \implies J = -\frac{11}{24} \approx -1.7917
\]

e) \(T\) is a closed path and \(\vec{F}\) is conservative, so \(\int_T \vec{F} \cdot d\vec{r} = 0\). Let \(T_1\) be the line segment from \((1, 0, 0)\) to \((0, 1, 0)\), \(T_2\) be the line segment from \((0, 1, 0)\) to \((0, 0, 1)\) and \(T_3\) be the line segment from \((0, 0, 1)\) to \((1, 0, 0)\). On \(T_1\), \(z = 0\), so \(\int_{T_1} \hat{z} \cdot d\vec{r} = 0\). On \(T_2\), \(x = 0\), so \(\hat{i} \cdot d\vec{r} = 0\) and \(\int_{T_2} \hat{i} \cdot d\vec{r} = 0\). Parametrize \(T_3\) by \(\vec{r}(t) = t \hat{i} + (1 - t) \hat{k}\), \(0 \leq t \leq 1\). Then \(\frac{d\vec{r}}{dt} = \hat{i} - \hat{k}\) and
\[
\int_T (\hat{z} + \vec{F}) \cdot d\vec{r} = \int_{T_3} \hat{z} \cdot d\vec{r} = \int_0^1 t \hat{i} \cdot (\hat{i} - \hat{k}) \, dt = \frac{1}{2}
\]

3) Let \(R\) be the region in the first quadrant of the \(xy\)-plane bounded by the coordinate axes and the curve \(y = 1 - x^2\). Let \(C\) be the boundary of \(R\), oriented counterclockwise.

a) Evaluate \(\int_C x \, ds\).

b) Evaluate \(\int_C \vec{F} \cdot d\vec{r}\), where \(\vec{F}(x, y) = (\sin(x^2) - xy) \hat{i} + (x^2 + \cos(y^2)) \hat{j}\).

Solution. a) Let \(C_1\) be the line segment from \((0, 1)\) to \((0, 0)\), \(C_2\) be the line segment from \((0, 0)\) to \((1, 0)\) and \(C_3\) be the curve \(y = 1 - x^2\) from \((1, 0)\) to \((0, 1)\). Then
All together
\[ \int_C x \, ds = \frac{1}{2} + \frac{1}{12} \approx 1.3484 \]

b) By either Stokes’ Theorem or Green’s Theorem
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial}{\partial x} (x^2 + \cos(y^2)) - \frac{\partial}{\partial y} (\sin(x^2) - xy) \right) \, dx \, dy = \iint_R 3 \, dx \, dy \]
\begin{align*}
&= 3 \int_0^1 dx \int_0^{1-x^2} dy \, x = 3 \int_0^1 dx \, (1-x^2) \, x = 3 \left[ \frac{1}{2} - \frac{1}{3} \right] = 1/3
\end{align*}

4) Let \( S \) be the part of the surface \( x^2 + y^2 + 2z = 2 \) that lies above the square \(-1 \leq x \leq 1, -1 \leq y \leq 1\).
a) Find \( \iint_S \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} \, dS \).
b) Find the flux of \( \mathbf{F} = x\hat{i} + y\hat{j} + z\hat{k} \) upward through \( S \).

Solution. Let \( G(x, y, z) = x^2 + y^2 + 2z \). Then
\[ \hat{n} \, dS = \frac{\nabla G \times \mathbf{F}}{\sqrt{1+\hat{n} \cdot \nabla G}} \, dx \, dy = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{1+x^2+y^2}} \, dx \, dy \]
\[ \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} \, dS = \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} \, \sqrt{1+x^2+y^2} \, dx \, dy = (x^2+y^2) \, dx \, dy \]
\[ \mathbf{F} \cdot \hat{n} \, dS = [x\hat{i} + y\hat{j} + z\hat{k}] \cdot [x\hat{i} + y\hat{j} + z\hat{k}] \, dx \, dy = [x^2 + y^2 + z] \, dx \, dy = \left[ 1 + \frac{1}{2}(x^2 + y^2) \right] \, dx \, dy \]
since \( z = 1 - \frac{1}{2}(x^2 + y^2) \) on \( S \).
\[ a) \quad \iint_S \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} \, dS = \iint_S (x^2+2y^2) \, dx \, dy = 4 \int_0^1 dx \int_0^1 dy \, (x^2+y^2) = 4 \int_0^1 dx \, (x^2 + \frac{1}{3}) = \frac{3}{3} \]
b) \[ \iint_S \mathbf{F} \cdot \hat{n} \, dS = \int_0^1 dx \int_0^1 dy \left[ 1 + \frac{1}{2}(x^2 + y^2) \right] = 2 \times 2 + \frac{1}{2} \times \frac{3}{2} = \frac{10}{3} \]

5) A smooth surface \( S \) lies above the plane \( z = 0 \) and has as its boundary the circle \( x^2 + y^2 = 4y \) in the plane \( z = 0 \). This circle also bounds a disk \( D \) in that plane. The volume of the 3–dimensional region \( R \) bounded by \( S \) and \( D \) is 10 cubic units. Find the flux of \( \mathbf{F} = (x+yz)\hat{i} + (y+xy^2)\hat{j} + (z+2x+3y)\hat{k} \) through \( S \) in the direction outward from \( R \).

Solution. On the bottom surface, \( z = 0 \) and the outward normal is \(-\hat{k}\), so that
\[ \iint_D \mathbf{F} \cdot \hat{n} \, dS = -\iint_D \mathbf{F} \cdot \hat{k} \, dx \, dy = -\iint_D (2x+3y) \, dx \, dy \]
The disk \( D \) has radius 2, area \( 4\pi \) and is centred on \((0, 2)\). So
\[ \iint_D \mathbf{F} \cdot \hat{n} \, dS = -4\pi(2x+3y) = -4\pi(2 \times 0 + 3 \times 2) = -24\pi \]
As \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x+y^2z) + \frac{\partial}{\partial y}(y+xy^2) + \frac{\partial}{\partial z}(z+2x+3y) = (1+2xy)+(1-2xy)+(1) = 3 \) the divergence theorem gives
\[ \iint_S \mathbf{F} \cdot \hat{n} \, dS = \iint_R \nabla \cdot \mathbf{F} \, dV - \iint_D \mathbf{F} \cdot \hat{n} \, dS \]
\[ = \iint_R 3 \, dV - (-24\pi) = 3 \times 10 + 24\pi = \frac{30+24\pi}{3} \]

6) An object moves along a curve in the \( xy \)–plane having polar equation \( r = \frac{1}{\cos \alpha} \) (where \( \alpha \) is a constant) under the influence of a central force so that the object has no transverse acceleration.
a) Verify that \( r^2 \dot{\theta} = h \) remains constant as the object moves.
b) Express the magnitude of the acceleration of the object as a function of \( r \) and \( h \).

Solution. We did not cover this material this year.