MATHEMATICS 317 December 2000 Final Exam

[15] 1) A skier descends the hill \( z = \sqrt{4-x^2-y^2} \) along a trail with parameterization

\[
x = \sin(2\theta), \quad y = 1 - \cos(2\theta), \quad z = 2\cos\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

Let \( P \) denote the point on the trail where \( x = 1 \).
(a) Find Frenet frame \( \hat{T}, \hat{N}, \hat{B} \) and the curvature \( \kappa \) of the ski trail at the point \( P \).
(b) The skier’s acceleration at \( P \) is \( \vec{a} = (-2, 3, -2\sqrt{2}) \). Find, at \( P \),
   (i) the rate of change of the skier’s speed and
   (ii) the skier’s velocity (a vector).

[15] 2) Consider the following force field, in which \( m, n, p, q \) are constants:

\[
\vec{F} = (mxyz + z^2 - ny^2) \hat{i} + (x^2z - 4xy) \hat{j} + (x^2y + pxz + qz^3) \hat{k}
\]

(a) Find all values of \( m, n, p, q \) such that \( \oint \vec{F} \cdot d\vec{r} = 0 \) for all piecewise smooth closed curves \( C \) in \( \mathbb{R}^3 \).
(b) For every possible choice of \( m, n, p, q \) in (a), find the work done by \( \vec{F} \) in moving a particle from the bottom to the top of the sphere \( x^2 + y^2 + z^2 = 2z \). (The direction of \( \hat{k} \) defines “up”.)

[15] 3) Let \( a \) and \( b \) be positive constants, and let \( S \) be the part of the conical surface

\[
a^2z^2 = b^2(x^2 + y^2)
\]

where \( 0 \leq z \leq b \). Consider the surface integral

\[
I = \iint_S (x^2 + y^2) dS.
\]

(a) Use projection to express \( I \) as a double integral over a disk in the \( xy \)-plane.
(b) Use the parametrization \( x = t \cos \theta, \ y = t \sin \theta, \) etc., to express \( I \) as a double integral over a suitable region in the \( t\theta \)-plane.
(c) Evaluate \( I \) using the method of your choice.

[15] 4) Let \( S \) be the curved surface below, oriented by the outward normal:

\[
x^2 + y^2 + 2(z - 1)^2 = 6, \quad z \geq 0.
\]

(E.g., at the high point of the surface, the unit normal is \( \hat{k} \).) Define

\[
\hat{G} = \nabla \times \vec{F}, \quad \text{where} \quad \vec{F} = (xz - y^3 \cos z) \hat{i} + x^3e^z \hat{j} + yze^{-x^2+y^2+z^2} \hat{k}.
\]

Find \( \iint_S \hat{G} \cdot d\hat{S} \).

[15] 5) Let \( R \) be the part of the solid cylinder \( x^2 + (y - 1)^2 \leq 1 \) satisfying \( 0 \leq z \leq y^2 \); let \( S \) be the boundary of \( R \).

Given \( \vec{F} = x^2 \hat{i} + 2y \hat{j} - 2z \hat{k} \),
(a) Find the total flux of \( \vec{F} \) outward through \( S \).
(b) Find the total flux of \( \vec{F} \) outward through the (vertical) cylindrical sides of \( S \).

Hint: \( \int_0^\pi \sin^n \theta \ d\theta = \frac{n-1}{n} \int_0^\pi \sin^{n-2} \theta \ d\theta \) for \( n = 2, 3, 4, \ldots \)

[15] 6) Three quickies (five marks each):

(a) A moving particle has velocity and acceleration vectors that satisfy \( |\vec{v}| = 1 \) and \( |\vec{a}| = 1 \) at all times. Prove that the curvature of this particle’s path is a constant; evaluate this constant.
(b) A moving particle with position \( \vec{r}(t) = (x(t), y(t), z(t)) \) satisfies

\[
\vec{a} = f(\vec{r}, \vec{v}) \vec{r}
\]
for some scalar-valued function $f$. Prove that $\mathbf{r} \times \mathbf{v}$ is constant.

(c) Calculate $\int \int_{S} (x \mathbf{i} - y \mathbf{j} + z^2 \mathbf{k}) \cdot d\mathbf{S}$, where $S$ is the boundary of any solid right circular cylinder of radius $b$ with one base in the plane $z = 1$ and the other base in the plane $z = 3$.

[10] 7) Let $u = u(x, y, z)$ be a solution of Laplace’s Equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a smooth simply connected region $\mathcal{R}$ of $\mathbb{R}^3$.

(a) Prove that the total flux of $\mathbf{F} = \nabla u$ out through the boundary of $\mathcal{R}$ is zero.

(b) Prove that the total flux of $\mathbf{G} = u \nabla u$ out through the boundary of $\mathcal{R}$ equals

$$\iiint_{\mathcal{R}} \left[ (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 \right] dV.$$