Quiz 3 math 200-106

Let \( f(x, y) = (y - x^2)(y - 1) \)

1. Find all Critical Points
2. Classify each critical point as either a local maximum, a local minimum, a saddle point, or not an ordinary critical point.
3. Find the coordinates of the points where \( f \) is maximum and where \( f \) is minimum subject to the constraint \( x^2 + y^2 = 2 \).
4. Find the coordinates of the maximum and minimum of \( f \) on the domain \( \mathcal{D}(x, y) \) \( x^2 + y^2 \leq 2 \).
5. Sketch the contour plot of \( f(x, y) \).
   
   Hint: Start by drawing the contour \( f(x, y) = 0 \).
   
   Make sure your sketch is consistent with your answers for parts 1 - 4.
Solutions

\[ f(x,y) = (y-x^2)(y-1) \]

\[ f_x = -2x(y-1) = 0 \quad \text{(i)} \]
\[ f_y = (y-1) + (y-x^2) = 2y-1-x^2 = 0 \quad \text{(ii)} \]

\[ \Rightarrow \quad x = 0 \quad \text{or} \quad y = 1 \]

Case 1: \( x = 0 \) then \( \text{(ii)} \Rightarrow 2y-1 = 0 \Rightarrow y = \frac{1}{2} \)
So \((0, \frac{1}{2})\) is a crit pt.

Case 2: \( y = 1 \) then \( \text{(ii)} \Rightarrow 2-1-x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1 \) or \( x = -1 \)
So \((1,1)\) and \((-1,1)\) are also crit. pts

Critical Points are \((0, \frac{1}{2})\) \((1,1)\) \((-1,1)\)

\[ f_{xx} = -2(y-1) \quad f_{yy} = 2 \quad f_{yx} = -2x \quad f_{yx} = 4x^2 \]

\[ D = f_{xx} f_{yy} - f_{yx}^2 = -4(y-1) - 4x^2 \]

\[ D(0, \frac{1}{2}) = -4(\frac{1}{2}) = -2 < 0 \Rightarrow (0, \frac{1}{2}) \text{ local minimum} \]

\[ D(1,1) = -4 = D(-1,1) = -4 < 0 \Rightarrow (\pm 1, 1) \text{ are saddle points} \]

**g(x,y) = x^2 + y^2 - z**

\[ f_x = \lambda g_x \quad -2x(y-1) = \lambda 2x \quad \text{(i)} \]
\[ f_y = \lambda g_y \quad 2y-1-x^2 = \lambda 2y \quad \text{(ii)} \]
\[ g = 0 \quad x^2 + y^2 = 2 \quad \text{(iii)} \]
\[ c \Rightarrow x = 0 \text{ or } \lambda = -(y-1) \]

**Case 1** \( x = 0 \) then \( \circled{\text{i}} \Rightarrow 2y - 1 = \lambda 2y \)
\[ \Rightarrow x = 1 - \frac{1}{2y} = 1 - \frac{1}{\pm \sqrt{2}} \]

so \( (0, \pm \sqrt{2}), (0, -\sqrt{2}) \) are possible max/mins

**Case 2** \( \lambda = -y + 1 \) then \( \circled{\text{ii}} \Rightarrow 2y - 1 - x^2 = 2y(1 - y) \)
\[ \Rightarrow x^2 = y^2 - 2 \]
\[ \Rightarrow 2y - 1 + y^2 - 2 = 2y - 2y \]
\[ \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1 \]
then \( \circled{\text{iii}} \Rightarrow x = \pm 1 \)

so \( (1,1), (1,-1), (-1,1), (-1,-1) \) are possible max/mins

\[
\begin{array}{c|c}
(x, y) & f(x,y) = (y-x^2)(y-1) \\
\hline
(0, \sqrt{2}) & \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2} \\
(0, -\sqrt{2}) & -\sqrt{2}(-\sqrt{2} - 1) = 2 + \sqrt{2} \\
(\pm 1, 1) & (1 - (\pm 1)^2)(1 - 1) = 0 \\
(\pm 1, -1) & (-1 - (\pm 1)^2)(-1 - 1) = -2(-2) = 4 
\end{array}
\]

\( \pm 1, 1 \) are mins with value 0
\( \pm 1, -1 \) are maxs with value 4

\( \text{Answer to part (3)} \)

Max/Min of \( f \) on \( x^2 + y^2 \leq 1 \) must occur at either a critical point local max or local min on the interior or on the boundary i.e at \((0, \frac{1}{2}), (\pm 1, 1), \text{ or } (\pm 1, -1)\)
\[ f(0, \frac{1}{2}) = \frac{1}{2} \left(-\frac{1}{2}\right) = -\frac{1}{4} \] smallest value so

\[ (0, \frac{1}{2}) \text{ is a global min with value } -\frac{1}{4} \]

\[ (1, -1) \quad (-1, -1) \text{ are global maxs with value } 4 \]

\[ f(x, y) = (y - x^2)(y - 1) \]

Note:
- Contours are tangent to \( x^2 + y^2 = 2 \) at \((\pm 1, -1) \) and \((0, \pm \sqrt{2})\)
- Saddle points look like distorted \( x \)s
- Local min looks like \( \infty \)
- \( f(x, y) = 0 \) is the union of the parabola \( y = x^2 \) and the line \( y = 1 \).