Quiz 3 math 200-106

Let $f(x,y) = (y-x^2)(y-1)$

① Find all Critical Points

② Classify each critical point as either a local maximum, a local minimum, a saddle point, or not an ordinary critical point.

③ Find the coordinates of the points where f is maximum and where f is minimum subject to the constraint $x^2+y^2=2$

④ Find the coordinates of the maximum and minimum of f on the domain $\{(x,y) \quad x^2+y^2 \le 2\}$

⑤ Sketch the contour plot of $f(x,y)$.

Hint: Start by drawing the contour $f(x,y)=0$

Make sure your sketch is consistant with your answers for parts ① - ④.

Solutions

$f(x,y) = (y-x^2)(y-1)$

$f_x = -2x(y-1) = 0$ (a)

$f_y = (y-1) + (y-x^2) = 2y-1-x^2 = 0$ (b)

(a) $\Rightarrow$ $x=0$ or $y=1$

Case 1: $x=0$ then (b) $\Rightarrow$ $2y-1=0 \Rightarrow y=\frac{1}{2}$

so $(0,\frac{1}{2})$ is a crit pt.

Case 2: $y=1$ then (b) $\Rightarrow$ $2-1-x^2=0 \Rightarrow x^2=1 \Rightarrow x=1$ or $x=-1$

so $(1,1)$ and $(-1,1)$ are also crit. pts

$\boxed{\text{Critical Points are } (0,\tfrac{1}{2})\ (1,1)\ (-1,1)}$

$f_{xx} = -2(y-1)$ $\quad f_{yy}=2$ $\quad f_{yx} = -2x$ $\quad f_{yx}^2 = 4x^2$

$D = f_{xx}f_{yy} - f_{xy}^2 = -4(y-1) - 4x^2$

$D(0,\frac{1}{2}) = -4(-\frac{1}{2}) = 2>0$ $\quad f_{yy}=2>0$ $\Rightarrow$ $\boxed{(0,\tfrac{1}{2}) \text{ local minimum}}$

$D(1,1) = -4 = D(-1,1) = -4 < 0 \Rightarrow$ $\boxed{(\pm1,1) \text{ are saddle points}}$

$g(x,y) = x^2+y^2-2$

$f_x = \lambda g_x$ $\qquad -2x(y-1) = \lambda 2x$ (i)

$f_y = \lambda g_y$ $\qquad 2y-1-x^2 = \lambda 2y$ (ii)

$g=0$ $\qquad x^2+y^2=2$ (iii)

$\int$

(i) $\Rightarrow$ $x = 0$ or $\lambda = -(y-1)$

Case 1 $x=0$ then (ii) $\Rightarrow$ $2y-1 = \lambda 2y$
(iii) $\Rightarrow$ $y = \pm\sqrt{2}$ $\Big\} \Rightarrow \lambda = 1 - \frac{1}{2y} = 1 - \frac{1}{\pm 2\sqrt{2}}$

so $(0,\sqrt{2})$, $(0,-\sqrt{2})$ are possible max/min

Case 2 $\lambda = -y+1$ then (ii) $\Rightarrow$ $2y-1-x^2 = 2y(1-y)$
(iii) $\Rightarrow$ $-x^2 = y^2-2$

$\Rightarrow \cancel{2y} - 1 + y^2 - 2 = \cancel{2y} - 2y^2 \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1$
then (iii) $\Rightarrow$ $x = \pm 1$

so $(1,1), (1,-1), (-1,1), (-1,-1)$ are possible max/mins.

(x,y)	$f(x,y) = (y-x^2)(y-1)$	
$(0,\sqrt{2})$	$\sqrt{2}(\sqrt{2}-1) = 2-\sqrt{2}$	
$(0,-\sqrt{2})$	$-\sqrt{2}(-\sqrt{2}-1) = 2+\sqrt{2}$	
$(\pm1, 1)$	$(1-(\pm1)^2)(1-1) = 0$	$\leftarrow$ mins
$(\pm1,-1)$	$(-1-(\pm1)^2)(-1-1) = (-2)(-2) = 4$	$\leftarrow$ maxs

$$\boxed{\begin{array}{l}(\pm1, 1) \text{ are mins with value } 0 \\ (\pm1,-1) \text{ are maxs with value } 4\end{array}}$$

$\leftarrow$ Answer to part (3)

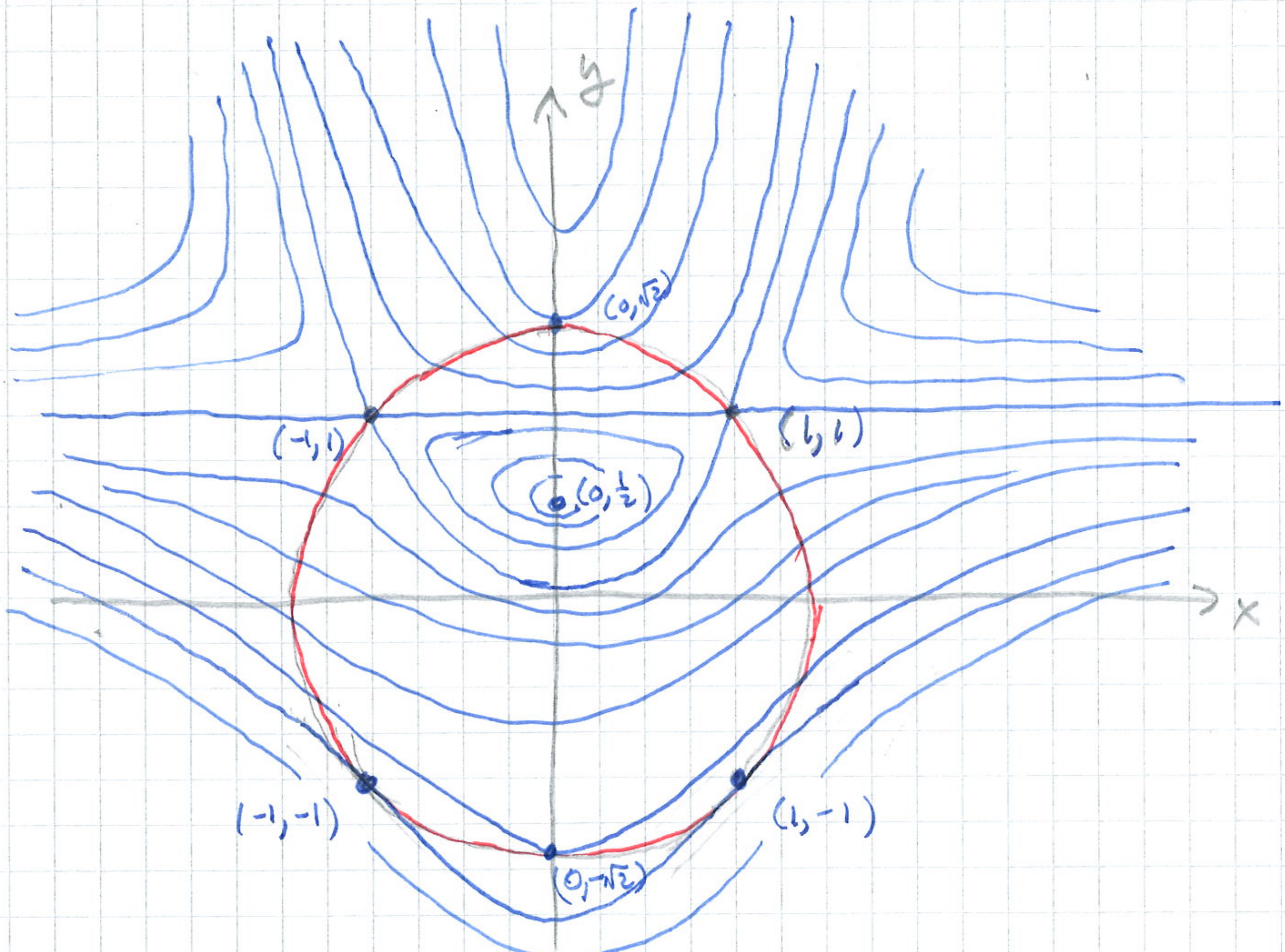
Max/Min of f on $x^2+y^2 \le 1$ must occur at either a ~~critical point~~ local max or local min on the interior or on the boundary i.e at $(0,\frac{1}{2})$, $(\pm1,1)$, or $(\pm1,-1)$

$f(0, \frac{1}{2}) = \frac{1}{2}(-\frac{1}{2}) = -\frac{1}{4}$ smallest value so

$(0, \frac{1}{2})$ is a global min with value $-\frac{1}{4}$

$(1,-1)$ $(-1,-1)$ are global maxs with value 4

$f(x,y) = (y-x^2)(y-1)$



Note:

- contours are tangent to $x^2+y^2=2$ at $(\pm 1,-1)$ & $(0,\pm\sqrt{2})$
- saddle points look like distorted Xs : 
- local min looks like 
- $f(x,y)=0$ is the union of the parabola $y=x^2$ and the line $y=1$.