

THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253

Midterm 2

13 November 2013

TIME: 50 MINUTES

LAST NAME: Solutions FIRST NAME: _____

STUDENT # : _____ INSTRUCTOR'S NAME: _____

This Examination paper consists of 9 pages (including this one). Make sure you have all 9.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1		8
Q2		8
Q3		10
Q4		9
TOTAL		35

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Sabin Cautis, Lior Silberman

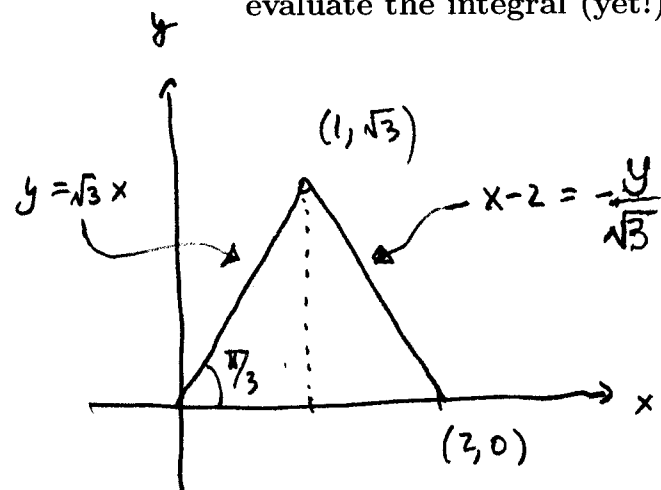
Q1 [8 points]

Consider the integral

$$\iint_T \sqrt{3} dA$$

where T is the triangle in the xy -plane with vertices $(0,0)$, $(2,0)$, and $(1,\sqrt{3})$.

- (a) [2 points] Write the integral as an iterated integral where you integrate x first. Do not evaluate the integral (yet!).

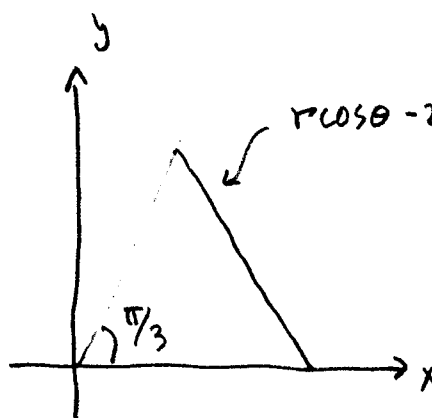


$$\int_{y=0}^{\sqrt{3}} \int_{x=\frac{y}{\sqrt{3}}}^{2-\frac{y}{\sqrt{3}}} \sqrt{3} dx dy$$

- (b) [2 points] Write the integral as an iterated integral where you integrate y first. Hint: you may write this integral as the sum of two integrals. Do not evaluate the integral (yet!).

$$\iint_T \sqrt{3} dA = \int_{x=0}^1 \int_{y=0}^{\sqrt{3}x} \sqrt{3} dy dx + \int_{x=1}^2 \int_{y=0}^{\sqrt{3}(2-x)} \sqrt{3} dy dx$$

- (c) [2 points] Write the integral as an iterated integral in polar coordinates. **Do not evaluate the integral (yet!).**



$$r \cos \theta - 2 = -\frac{1}{\sqrt{3}} r \sin \theta$$

$$r \left(\cos \theta + \frac{1}{\sqrt{3}} \sin \theta \right) = 2$$

$$r = \frac{2}{\cos \theta + \frac{1}{\sqrt{3}} \sin \theta}$$

$$\iint_T \sqrt{3} dA = \int_{\theta=0}^{\pi/3} \int_{r=0}^{\frac{2}{\cos \theta + \frac{1}{\sqrt{3}} \sin \theta}} \sqrt{3} r dr d\theta$$

- (d) [2 points] Evaluate the integral using any method.

Easiest: $\iint_T \sqrt{3} dA = \sqrt{3} \text{Area}(T) = \sqrt{3} \left(\frac{1}{2} \cdot 2 \cdot \sqrt{3} \right) = 3$

Integration in part (a):

$$\int_{y=0}^{\sqrt{3}} \int_{x=\frac{y}{\sqrt{3}}}^{2-\frac{1}{\sqrt{3}}y} \sqrt{3} dx dy = \sqrt{3} \int_{y=0}^{\sqrt{3}} \left(2 - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}y \right) dy$$

$$= 2\sqrt{3} \int_0^{\sqrt{3}} \left(1 - \frac{1}{\sqrt{3}}y \right) dy = 2\sqrt{3} \left[y - \frac{1}{2\sqrt{3}}y^2 \right]_0^{\sqrt{3}}$$

$$= 2\sqrt{3} \left[\sqrt{3} - \frac{1}{2\sqrt{3}}(\sqrt{3})^2 \right] = 2\sqrt{3} \left(\frac{\sqrt{3}}{2} \right) = 3.$$

Q2 [8 points]

Consider the following iterated integral.

$$\int_{x=0}^1 \int_{y=1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} x \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy dx$$

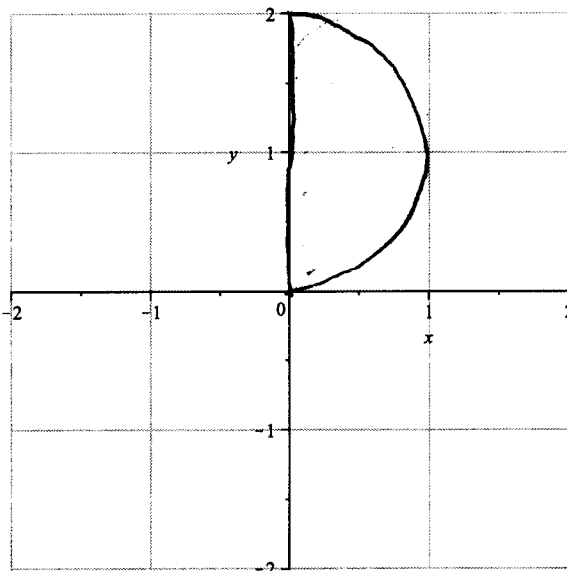
(a) [3 points] Sketch the domain of integration on the graph provided below.

$$y = 1 \pm \sqrt{1-x^2}$$

$$\pm\sqrt{1-x^2} = 1-y$$

$$\Rightarrow 1-x^2 = (1-y)^2$$

$$\Rightarrow (y-1)^2 + x^2 = 1$$



(b) [5 points] Compute the integral.

Switch order of integration:

$$x = \sqrt{1-(y-1)^2} = \sqrt{2y-y^2}$$

$$\begin{aligned} y=0 &\leadsto u=\pi \\ y=2 &\leadsto u=\pi\left(1-4+\frac{8}{3}\right) \\ &= -\frac{\pi}{3} \end{aligned}$$

$$\int_{y=0}^2 \int_{x=0}^{\sqrt{2y-y^2}} x \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dx dy$$

$$= \int_{y=0}^2 \left[\frac{x^2}{2} \right]_0^{\sqrt{2y-y^2}} \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy = \int_{y=0}^2 \left(y - \frac{1}{2}y^2\right) \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy$$

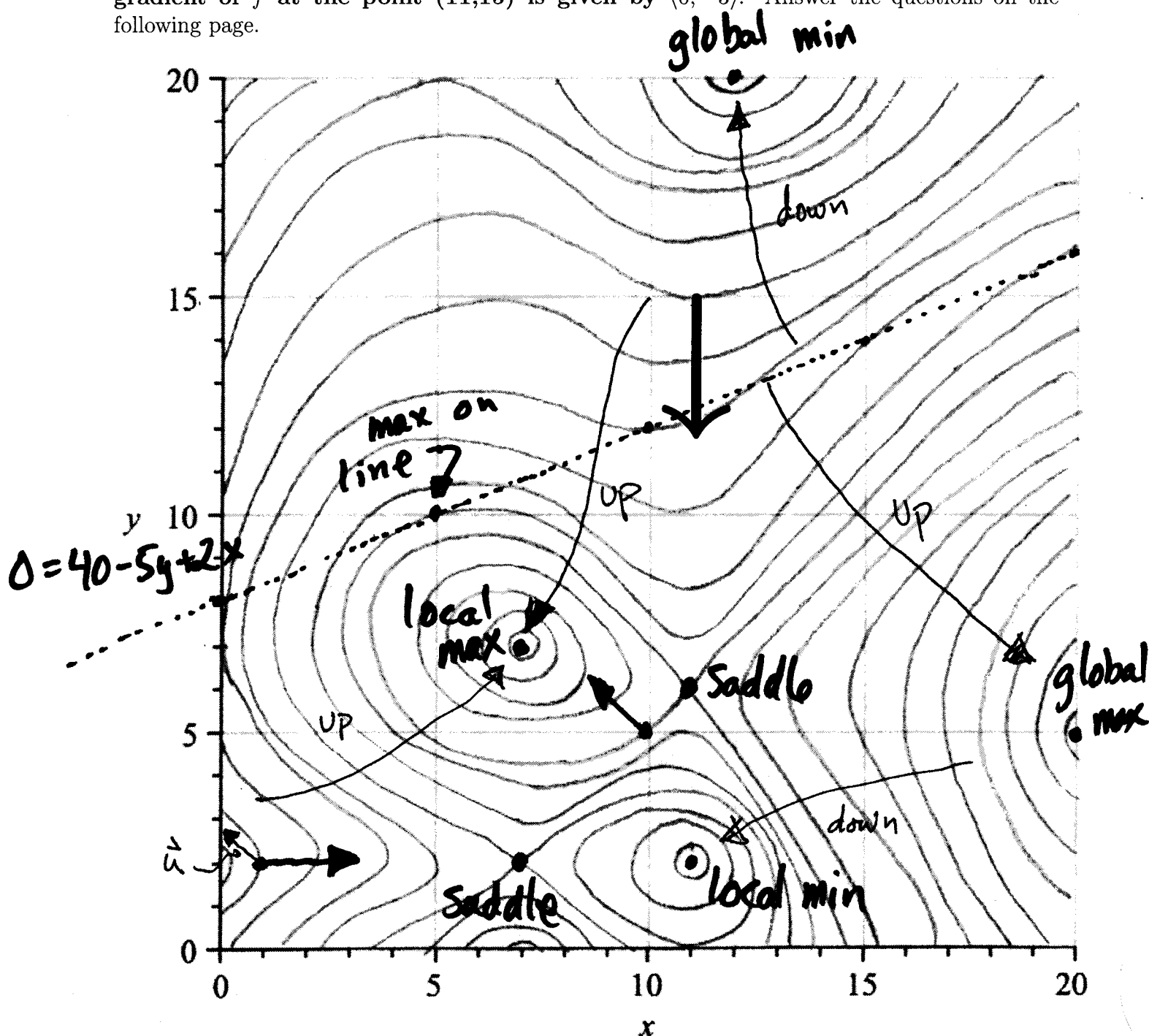
$$u = \pi\left(1-y^2+\frac{y^3}{3}\right)$$

$$du = (-2\pi y + \pi y^2) dy$$

$$\begin{aligned} &= \int_{u=\pi}^{-\pi/3} -\frac{1}{2\pi} \sin u du = \frac{1}{2\pi} \cos u \Big|_{\pi}^{-\pi/3} \\ &= \frac{1}{2\pi} \left[+\frac{1}{2} - (-1) \right] = \boxed{\frac{3}{4\pi}} \end{aligned}$$

Q3 [10 points]

The following is the contour plot of a function $f(x,y)$ with domain $\{(x,y) : 0 \leq x \leq 20, 0 \leq y \leq 20\}$. The values of the contours are spaced evenly. You may make reasonable assumptions about the function: the gradient does not vanish along an entire contour, the function does not fluctuate wildly on a scale smaller than that shown by the contours. As indicated below, The gradient of f at the point $(11,15)$ is given by $\langle 0, -3 \rangle$. Answer the questions on the following page.



- (a) (4 points.) Find the coordinates of the critical points of f on the interior of its domain. Classify each critical point as a local maximum, a local minimum, a saddle point, or "other".

$(7, 7)$ local max $(7, 2)$ saddle point
 $(11, 2)$ local min $(11, 6)$ saddle point

- (b) (2 points.) Find the coordinates of the global maximum of f and the global minimum of f . Global max/min occurs at critical points or on boundary. Since contours are evenly spaced, we can count contour lines to determine max/min. $\text{global max} = (20, 5)$ $\text{global min} = (13, 20)$

- (c) (1 point.) Let $u = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. Is the value of $(D_u f)(1, 2)$ positive, negative, or zero? Circle the correct answer.

$\vec{u} \cdot (\vec{\nabla} f)(1, 2)$ is negative since the angle between them is obtuse

- (d) (1 point.) Is the value of $f_{xx}(7, 7)$ positive, negative, or zero? Circle the correct answer.

$(7, 7)$ is a local max so $f_{xx} < 0$ negative

- (e) (1 point.) The direction of the gradient of f at the point $(10, 5)$ is given by which of the following (circle the correct answer):

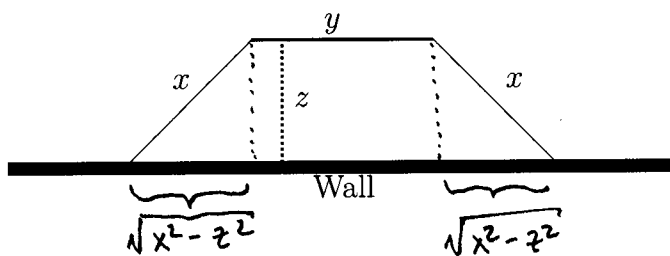
i) $\frac{1}{\sqrt{2}} \langle 1, 1 \rangle$
 ii) $\frac{1}{\sqrt{2}} \langle 1, -1 \rangle$
iii) $\frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ as shown in diagram
 iv) $\frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

- (f) (1 point.) Let $g(x, y) = 40 - 5y + 2x$. Find the approximate coordinates of the point which maximizes $f(x, y)$ subject to the constraint $g(x, y) = 0$.

maximum occurs on boundary or where $\vec{\nabla} f = \lambda \vec{\nabla} g$
 $\Leftrightarrow$ line is tangent to contour. This occurs
 at $(5, 10)$ which is the maximum

Q4 [9 points]

A trapezoidal enclosure is to be constructed by a fence with three sides and an existing wall. Two of the fence sides are the same length x and the third side is length y and is to be parallel to the existing wall. Let z be the distance from the wall to the parallel wall (see the picture). The area of the enclosure is required to be $3\sqrt{3}$ square meters and we wish to determine the fence which uses as little fencing material as possible. Find the values of the side lengths x and y and the distance z for the fence which has the minimal total length of fence.



$$\text{Area} = A(x, y, z) = z\sqrt{x^2 - z^2} + yz$$

$$\text{Length} = f(x, y, z) = 2x + y$$

$$\begin{array}{l} \boxed{\begin{array}{l} f_x = \lambda A_x \\ f_y = \lambda A_y \\ f_z = \lambda A_z \\ A = 3\sqrt{3} \end{array}} \Rightarrow \boxed{\begin{array}{l} 2 = \frac{\lambda z x}{\sqrt{x^2 - z^2}} \quad (1) \\ 1 = \lambda z \quad (2) \\ 0 = \lambda \left(\sqrt{x^2 - z^2} - \frac{z^2}{\sqrt{x^2 - z^2}} + y \right) \quad (3) \\ 3\sqrt{3} = z\sqrt{x^2 - z^2} + yz \quad (4) \end{array}} \quad (*) \end{array}$$

$$(1) \wedge (2) \Rightarrow 2 = \frac{x}{\sqrt{x^2 - z^2}} \Rightarrow 4(x^2 - z^2) = x^2 \Rightarrow 3x^2 = 4z^2 \Rightarrow \boxed{z = \frac{\sqrt{3}}{2}x} \quad (*)$$

substituting (*) into (3) and observing from (2) that $\lambda \neq 0$ we get

$$0 = \sqrt{x^2 - \frac{3x^2}{4}} - \frac{\frac{3x^2}{4}}{\sqrt{x^2 - \frac{3x^2}{4}}} + y \Rightarrow 0 = \frac{x}{2} - \frac{3x^2}{4} \left(\frac{2}{x} \right) + y$$

cont'd $\Rightarrow$

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$$0 = \frac{x}{2} - \frac{3x}{2} + y \Rightarrow \boxed{x=y} \text{ substituting this and (*) into}$$

$$(4) \text{ we get } 3\sqrt{3} = \frac{\sqrt{3}}{2} x \sqrt{x^2 - \frac{3}{4}x^2} + x \cdot \frac{\sqrt{3}}{2} x$$

$$\Rightarrow 3 = \frac{x}{2} \cdot \frac{x}{2} + \frac{x^2}{2} = \frac{3}{4}x^2 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

$$\text{so } \boxed{x=2, y=2, z=\sqrt{3}} \quad f(x,y,z) = 6$$

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$$0 = \frac{x}{2} - \frac{3x}{2} + y \Rightarrow \boxed{x=y} \text{ substituting this and (*) into}$$

$$\textcircled{4} \text{ we get } 3\sqrt{3} = \frac{\sqrt{3}}{2} x \sqrt{x^2 - \frac{3}{4}x^2} + x \cdot \frac{\sqrt{3}}{2} x$$

$$\Rightarrow 3 = \frac{x}{2} \cdot \frac{x}{2} + \frac{x^2}{2} = \frac{3}{4}x^2 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

$$\text{so } \boxed{x=2, y=2, z=\sqrt{3}} \quad f(x,y,z) = 6$$

boundary cases: $y=0$ (enclosed region is a triangle)
 $x=z$ (enclosed region is a rectangle).

each case is solved as a Lagrange multiplier problem with one fewer variable. We verify below that the minimum of f does not occur on the boundary

$$\underline{y=0}: \quad 2 = \frac{\lambda z x}{\sqrt{x^2 - z^2}} \quad (1), \quad 0 = \lambda \left(\sqrt{x^2 - z^2} - \frac{z^2}{\sqrt{x^2 - z^2}} \right) \quad (2), \quad 3\sqrt{3} = 2\sqrt{x^2 - z^2} \quad (3)$$

$\textcircled{1} \Rightarrow \lambda \neq 0$ substituting $\sqrt{x^2 - z^2} = \frac{3\sqrt{3}}{2}$ into $\textcircled{2}$ we get

$$0 = \frac{3\sqrt{3}}{2} - \frac{z^3}{3\sqrt{3}} \Rightarrow (3\sqrt{3})^2 = z^4 \Rightarrow z = \sqrt{3\sqrt{3}} \Rightarrow \sqrt{3\sqrt{3}} \sqrt{x^2 - 3\sqrt{3}} = 3\sqrt{3}$$

$$\Rightarrow x^2 - 3\sqrt{3} = 3\sqrt{3} \Rightarrow x = \sqrt{2}\sqrt{3\sqrt{3}} \Rightarrow f = 2\sqrt{2}\sqrt{3\sqrt{3}} \text{ which is bigger}$$

$$\text{than } 6 \text{ since } (2\sqrt{2}\sqrt{3\sqrt{3}})^4 = \frac{16 \cdot 4 \cdot 9 \cdot 3}{4 \cdot 2^4 \cdot 3^3} > \frac{6^4}{3 \cdot 2^4 \cdot 3^3}$$

$$\underline{x=z} \quad A = yz = 3\sqrt{3} \quad f(y,z) = 2z + y$$

$$\left. \begin{aligned} f_z = \lambda A_z &\Rightarrow 2 = \lambda y \\ f_y = \lambda A_y &\Rightarrow 1 = \lambda z \end{aligned} \right\} \Rightarrow 2 = \frac{y}{z}, \quad yz = 3\sqrt{3} \Rightarrow 2z^2 = 3\sqrt{3} \Rightarrow z = \frac{1}{\sqrt{2}}\sqrt{3\sqrt{3}} \\ y = \sqrt{2}\sqrt{3\sqrt{3}}$$

$$\hookrightarrow f = \frac{2}{\sqrt{2}}\sqrt{3\sqrt{3}} + \sqrt{2}\sqrt{3\sqrt{3}} = 2\sqrt{2}\sqrt{3\sqrt{3}} \text{ which as before is bigger than } 6$$