SECTION 106; MWF 3:00-4:00

The University of British Columbia

MATH 200 Final Examination -Dec 8, 2017

Closed book examination — Time: 150 minutes

Last Name Solutions **First** ______________________

Student Number ______________________ **Signature** ______________________

Special Instructions:

No memory aids, calculators, or electronic devices of any kind are allowed on the test. Where blanks are provided for answers, put your final answers in them. UNLESS OTHERWISE SPECIFIED, SHOW ALL YOUR WORK; little or no credit will be given for answers without the correct accompanying work. Numerical answers should be left in calculator-ready form, unless otherwise indicated. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1		9
2		9
3		8
4		12
5		14
6		13
7		9
8		10
9		6
Total		90

1.(a) (3pts) A function $z = f(x, y)$ is defined implicitly by $z^3 + xyz = 2$. Find f_x in terms of x, y, z.

$$3z^2 \frac{\partial z}{\partial x} + yz + xy\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x}\left(3z^2 + xy\right) = -yz$$

$$\boxed{f_x = \frac{\partial z}{\partial x} = \frac{-yz}{3z^2+xy}}$$

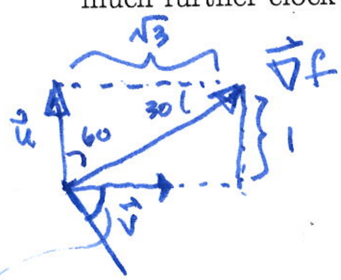
(b) (3pts) Find the equation of the tangent plane to the surface $z^3 + xy + x^3 = 3$ at the point $(1, 1, 1)$.

$$f(x,y,z) = z^3 + xy + x^3 \qquad \vec{\nabla} f = \langle 3x^2 + y,\ x,\ 3z^2 \rangle$$

$$(\vec{\nabla} f)(1,1,1) = \langle 4, 1, 3 \rangle$$

$$\boxed{4(x-1) + (y-1) + 3(z-1) = 0 \quad \text{or} \quad 4x + y + 3z = 8}$$

(c) (3pts) You stand at a point on the side of a hill. The slope of the ground in front of you is 1. You turn clockwise $\pi/2$ radians (90 degrees), and the slope of the ground in front is now $\sqrt{3}$. How much further clockwise should you turn so that the slope of the ground in front of you is zero?



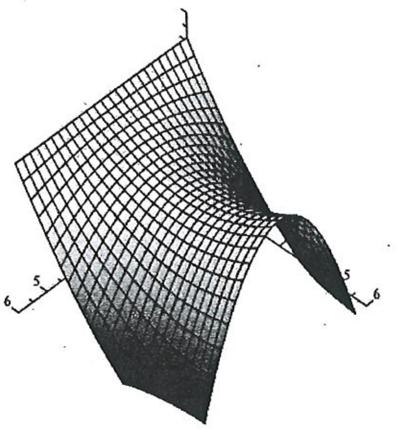
$$\vec{u} \cdot \vec{\nabla} f = 1$$
$$\vec{v} \cdot \vec{\nabla} f = \sqrt{3}$$

$\boxed{\text{turn } 60^\circ \text{ more, } \frac{\pi}{3} \text{ radians}}$

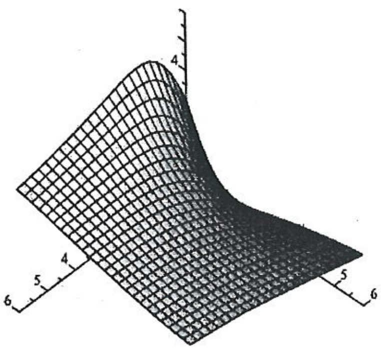
$\theta = 60^\circ$

ANSWER: You should turn another ____________________ (in radians) clockwise.

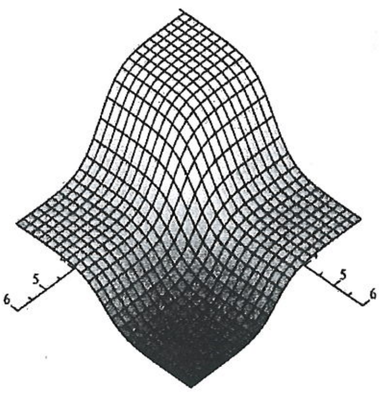
2. Consider the following contour plots labelled by a, b, and c and the following graphs labelled A, B, and C. The axes in the graphs are in the standard order: the positive z-axis is up, the positive x-axis is to the left, and the positive y-axis is to the right.



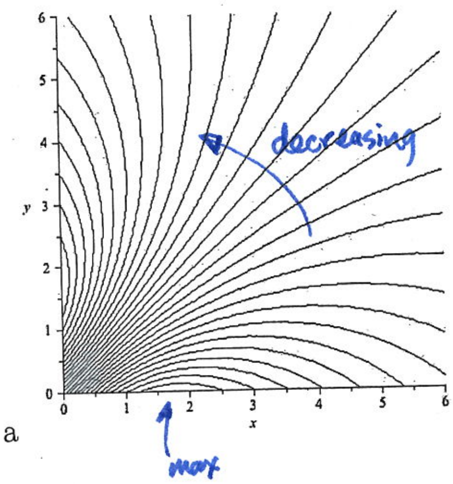
A



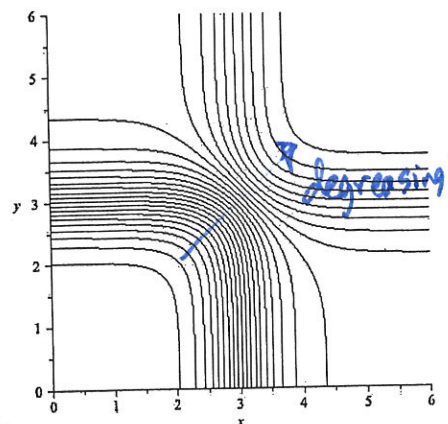
B



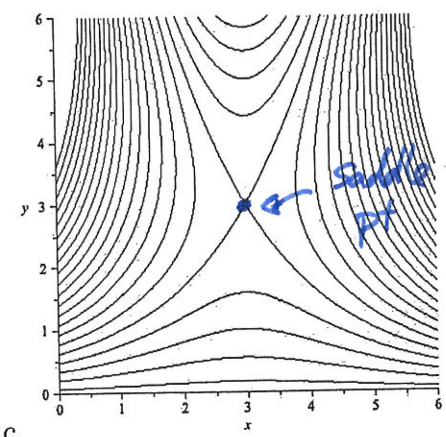
C



a



b



c

(3pts) Which of the following is the correct correspondence of contour plots to graphs?

1. $A = a$, $B = b$, $C = c$
2. $A = a$, $B = c$, $C = b$
3. $A = b$, $B = a$, $C = c$
4. $A = b$, $B = c$, $C = a$
5. $A = c$, $B = a$, $C = b$
6. $A = c$, $B = b$, $C = a$

(3pts) For each of the contour plots below, place the letter of the plot next to the correct statement. The approximate direction of gradient of the function at the point $(3,3)$ is...

- ...approximately in the northwest direction.
- ...approximately in the southwest direction. b
- ...approximately in the northeast direction.
- ...approximately in the southeast direction. a
- ...the zero vector. c

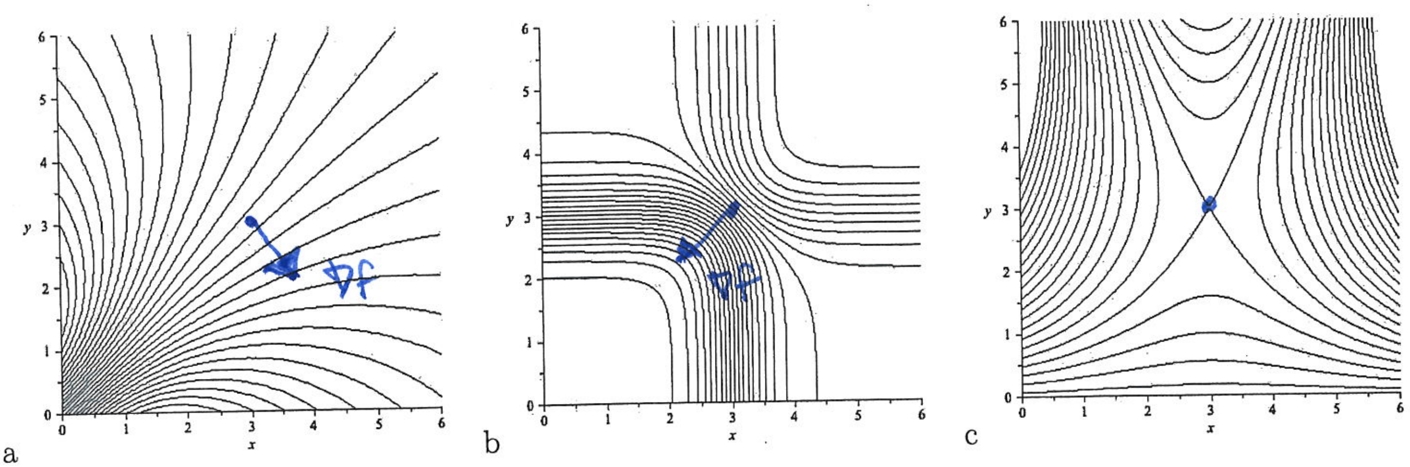
(3pts) For each of the contour plots below, circle the answer for each of the following .

1. The value of $f_x(3,3)$ in the plot (a) is
   - POSITIVE
   - NEGATIVE
   - ZERO

2. The value of $f_y(3,3)$ in the plot (b) is
   - POSITIVE
   - NEGATIVE
   - ZERO

3. The value of $f_{xx}f_{yy} - f_{xy}^2$ at the point (3,3) in the plot (c) is
   - POSITIVE
   - NEGATIVE
   - ZERO

saddle



a b c

3. (8 pts) The distance from the point $(3.03, 1.98, 5.99)$ to the origin $(0, 0, 0)$

is approximately ____________________ (use an appropriate linear approximation to fill blank with a sum of rational numbers (ie, fractions of integers)).

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $\quad f(3,2,6) = \sqrt{9+4+36} = 7$

$f(x,y,z) \approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6)$

$= 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6)$

$f(3.03, 1.98, 5.99) \approx 7 + \frac{3}{7}(.03) + \frac{2}{7}(-.02) + \frac{6}{7}(-.01)$

$= 7 + \frac{1}{7}(.09 - .04 - .06) = 7 + \frac{1}{7}(-.01)$

$$\boxed{= 7 - \frac{1}{700} = \frac{4899}{700}}$$

$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}$ at $(3,2,6)$ $\frac{3}{7}$

$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}$ " $\frac{2}{7}$

$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$ " $\frac{6}{7}$

or $7 + \frac{9}{700} - \frac{4}{700} - \frac{6}{700}$

4. The temperature in space is given by $T(x, y, z) = e^{x^2+3y^2+4z^2}$.

(a) (4 pts) In what direction at $Q = (1, 1, 1)$ is the temperature increasing most rapidly (express answer as a unit vector)?

$$\vec{\nabla} T = \langle 2xe^{x^2+3y^2+4z^2}, 6ye^{x^2+3y^2+4z^2}, 8ze^{x^2+3y^2+4z^2} \rangle$$

$$\vec{\nabla} T(1,1,1) = \langle 2e^8, 6e^8, 8e^8 \rangle \text{ has direction } \frac{\langle 1,3,4\rangle}{\sqrt{1+9+16}} = \frac{1}{\sqrt{26}}\langle 1,3,4\rangle$$

$$= 2e^8 \langle 1, 3, 4\rangle$$

(b) (3 pts) A bee starts at $Q = (1, 1, 1)$, then flies in the direction from (a) and continues along a straight line. Will the bee pass through the origin (explain)?

the line through $(1,1,1)$ in the direction of $\langle 1,3,4\rangle$ has parametric eqs

$$x = 1+t \qquad y = 1+3t \qquad z = 1+4t$$

however $0 = 1+t$, $0 = 1+3t$, $0 = 1+4t$ has no solutions so bee does not pass thru $(0,0,0)$

(c) (5 pts) The air pressure in space is given by $P(x, y, z) = 2x^2 + xy + 3y^2$. A fly starts at $R = (1, 2, 3)$, and t seconds later its position is $(x(t), y(t), z(t))$ where

$$x'(0) = x''(0) = \pi; \qquad y'(0) = y''(0) = \sqrt{7}; \qquad z'(0) = z''(0) = 0.$$

If $P(t)$ is the air pressure at $(x(t), y(t), z(t))$, find $\frac{d^2P}{dt^2}(0)$.

$$\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}$$

$$= (4x+y)\frac{dx}{dt} + (x+6y)\frac{dy}{dt} + 0$$

$$\frac{d^2P}{dt^2} = (4x'+y')x' + (4x+y)x'' + (x'+6y')y' + (x+6y)y''$$

$$\frac{d^2P}{dt^2}(0) = (4\pi+\sqrt{7})\pi + (4+2)\pi + (\pi+6\sqrt{7})\sqrt{7} + (1+12)\sqrt{7}$$

$$= 4\pi^2 + \sqrt{7}\pi + 6\pi + \sqrt{7}\pi + 42 + 13\sqrt{7}$$

$$= 4\pi^2 + (6+2\sqrt{7})\pi + 42 + 13\sqrt{7}$$

5. (a) [11pts] A mountain is represented by the graph of $f(x, y) = [x^2 + (y-1)^2]^6$ in space. Find the highest and lowest points on the mountian which lie directly over the ellipse $x^2 + 2y^2 = 18$ in the xy plane using Lagrange mutipliers (credit will not be given for any other method).

$g = x^2+2y^2-18 \qquad f = (x^2+(y-1)^2)^6$

$f_x = 12x(x^2+(y-1)^2)^5 = \lambda 2x \Rightarrow 6xy(x^2+(y-1)^2)^5 = \lambda xy$

$f_y = 12(y-1)(x^2+(y-1)^2)^5 = \lambda 4y \qquad 3x(y-1)(x^2+(y-1)^2)^5 = \lambda xy$

$\Rightarrow 6xy(x^2+(y-1)^2)^5 = 3x(y-1)(x^2+(y-1)^2)^5$

$x^2+(y-1)^2 \neq 0$ since that would mean $x=0$ $y=1$ and that is not on $x^2+2y^2=18$

so $6xy = 3x(y-1) \Rightarrow x=0$ or $2y=y-1 \Rightarrow y=-1$

$x=0$ $2y^2=18$ $\Rightarrow y=\pm 3$ $\qquad y=-1 \Rightarrow x^2+2=18 \Rightarrow x=\pm 4$

so max/min must occur at one of $(0,3)$ $(0,-3)$ $(4,-1)$ $(-4,-1)$

$f(0,3) = ((3-1)^2)^6 = 4^6$

$f(0,-3) = ((-3-1)^2)^6 = 4^{12} = (16)^6$

$f(4,-1) = (16+(-2)^2)^6 = (20)^6$

$f(-4,-1) = 20^6$

HEIGHT AT HIGHEST POINT: 20^6

HEIGHT AT LOWEST POINT: : 4^6

(b) [3pts] Find the highest and lowest points on the mountain which lie over the "solid ellipse" $x^2 + 2y^2 \leq 18$ in the xy plane.

other possible max/min are crit. pts on interior:

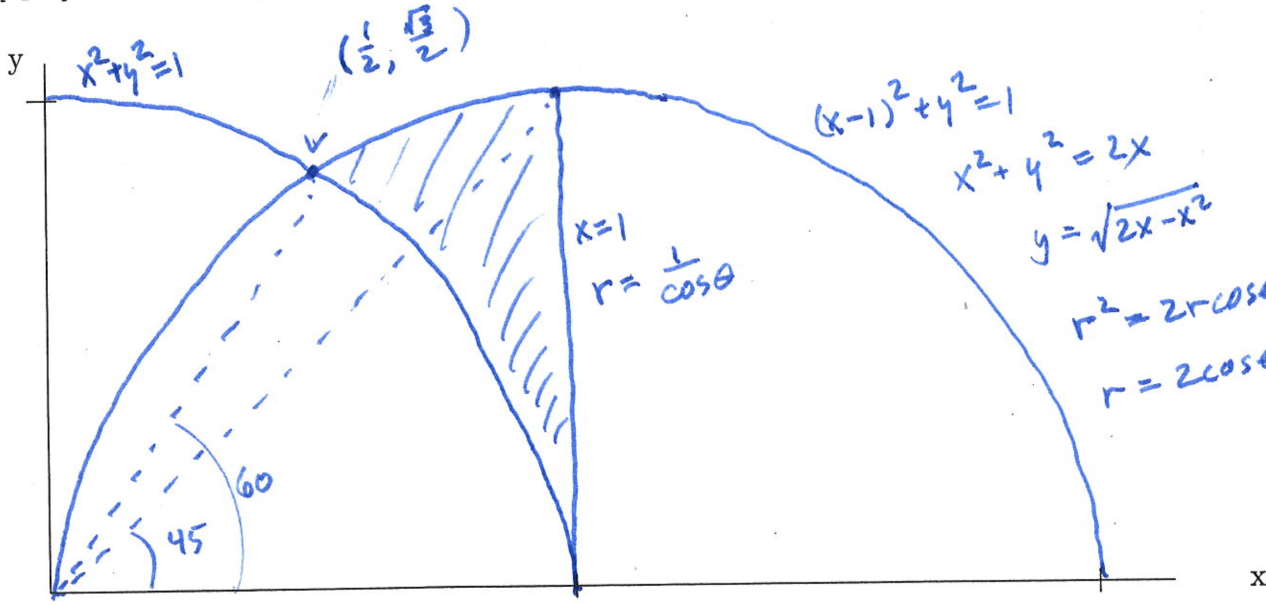
$12x(x^2+(y-1)^2)^5 = 0$
$12(y-1)(x^2+(y-1)^2)^5 = 0$ $\Rightarrow x=0$, $y=1$ $\qquad f(0,1)=0$

HEIGHT AT HIGHEST POINT: 20^6

HEIGHT AT LOWEST POINT: 0

6. Let R be the region in the POSITIVE QUADRANT of the xy plane and bounded by the curves $x = 1; x^2 + y^2 = 1; (x-1)^2 + y^2 = 1$

(a) [2pts] Shade the region R in the axis provided



(b) [8pts] Let A be the area of the region R. Fill in the blanks below (explanations not required)

(i) $A = \int_{1/2}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{2x-x^2}} (1)\, dy\, dx$ (use x, y coordinates)

(ii) $A = \int_{0}^{\pi/4} \int_{1}^{\sec\theta} (r)\, dr\, d\theta + \int_{\pi/4}^{\pi/3} \int_{1}^{2\cos\theta} (r)\, dr\, d\theta$

(use polar coordinates r, θ)

(c) [3pts] Calculate A from part (b) of the previous page using either of the double integrals from (i) or (ii).
(you may use the facts that $\frac{d}{d\theta}\tan(\theta) = \sec^2(\theta)$ and also $\cos^2(\theta) = (\cos(2\theta)+1)/2)$)

$$A = \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_1^{\sec\theta} d\theta + \int_{\pi/4}^{\pi/3} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left(\sec^2\theta - 1\right) d\theta + \int_{\pi/4}^{\pi/3} \frac{1}{2}\underbrace{\left(4\cos^2\theta - 1\right)}_{2\cos(2\theta)+2-1} d\theta$$

$$= \frac{1}{2}\left[\tan\theta - \theta\right]_0^{\pi/4} + \int_{\pi/4}^{\pi/3} \left(\cos(2\theta) + \frac{1}{2}\right) d\theta$$

$$= \frac{1}{2}\left[1 - \frac{\pi}{4}\right] + \left[\frac{1}{2}\sin(2\theta) + \frac{1}{2}\theta\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2} - \frac{\pi}{8} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{8}$$

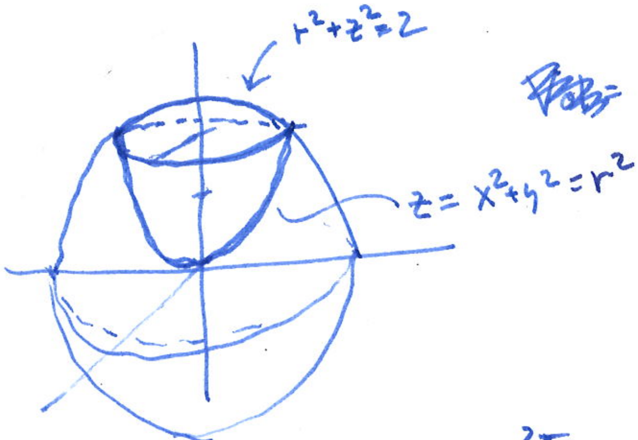
$$= \cancel{\frac{1}{2}} - \frac{\pi}{8} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \cancel{\frac{1}{2}} - \frac{\pi}{8}$$

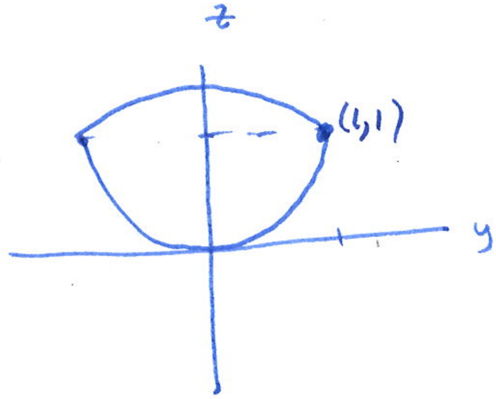
$$= \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \frac{\pi}{4} = \frac{\sqrt{3}}{4} + \frac{2\pi - 3\pi}{12}$$

$$= \frac{\sqrt{3}}{4} - \frac{\pi}{12}$$

7. (9pts). Consider the solid region in space which lies above the paraboloid $z = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 2$. Determine the volume V of this solid ~~and fill in the blanks below accordingly.~~

intersect at z where $z + z^2 = 2 \Rightarrow z^2 + z - 2 = 0 \quad (z+2)(z-1) = 0$





in cylindrical: $\quad \text{Vol} = \displaystyle\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$

$$= 2\pi \int_{r=0}^{1} r\,[z]_{r^2}^{\sqrt{2-r^2}}\,dr$$

$$= 2\pi \int_{r=0}^{1} (r\sqrt{2-r^2} - r^3)\,dr$$

$$= 2\pi \left[-\tfrac{1}{2}\cdot\tfrac{2}{3}(2-r^2)^{3/2} - \tfrac{1}{4}r^4\right]_0^1$$

$$= 2\pi\left(-\tfrac{1}{3} - \tfrac{1}{4} - \left[-\tfrac{1}{3}\,2^{3/2}\right]\right)$$

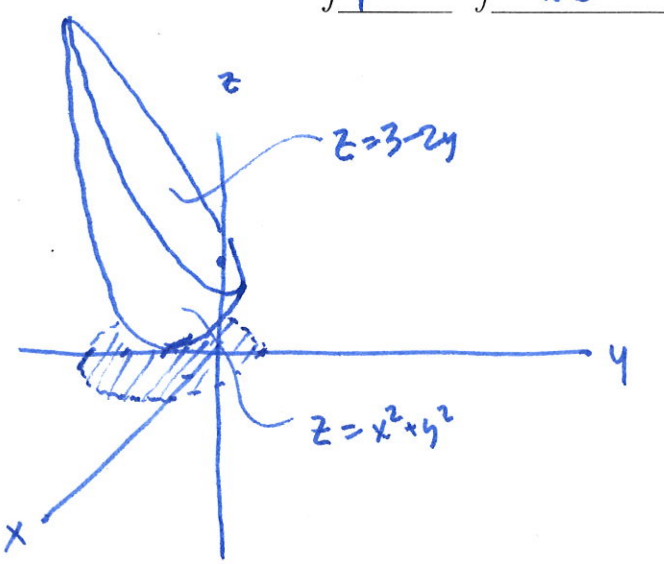
$$= 2\pi\left(-\tfrac{7}{12} + \tfrac{2\sqrt{2}}{3}\right) = \tfrac{\pi}{6}\left(8\sqrt{2} - 7\right)$$

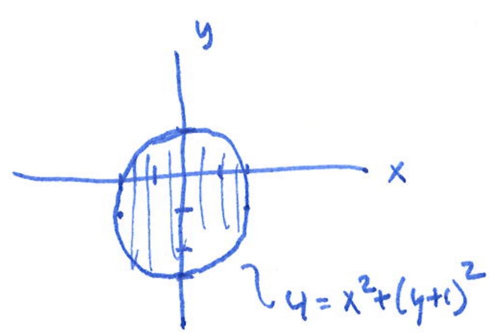
8. Let E be the region bounded by the plane $2y + z = 3$ and the paraboloid $z = x^2 + y^2$. Let $I = \iiint_E f(x, y, z)\,dV$. Fill in the following blanks below (No explanations required)

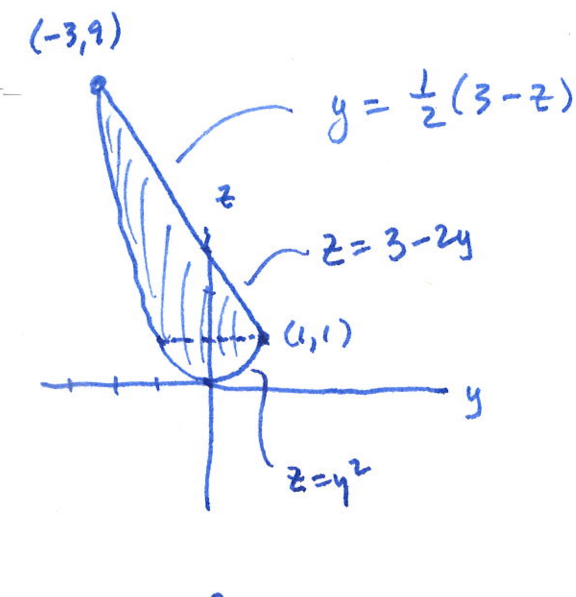
(a) (4pts) $I = \int_{-2}^{2} \int_{-\sqrt{4-(y+1)^2}}^{\sqrt{4-(y+1)^2}} \int_{x^2+y^2}^{3-2y} f(x, y, z)\,dz\,dx\,dy$

(b) (6pts) $I = \int_{0}^{1} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x, y, z)\,dx\,dy\,dz$

$+ \int_{1}^{9} \int_{-\sqrt{z}}^{\frac{1}{2}(3-z)} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x, y, z)\,dx\,dy\,dz$







$y^2+2y-3=0 \quad (y-1)(y+3)=0$

9. (a) (2pts) Sketch the ~~solid who's volume is~~ region of integration given by the integral

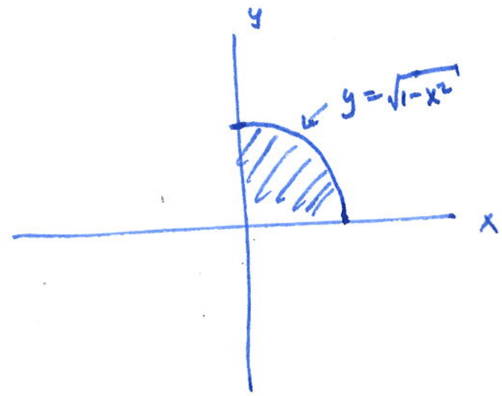
$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$$

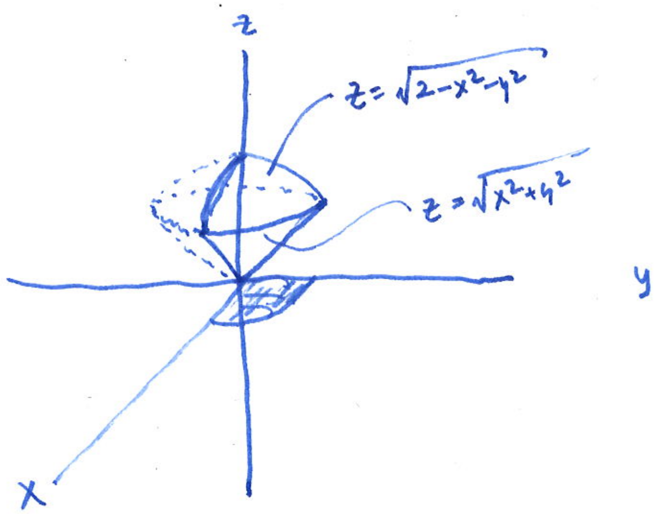
between $z = \sqrt{x^2+y^2}$ (cone)

and $z = \sqrt{2-x^2-y^2}$

(top part of sphere $x^2+y^2+z^2=2$)

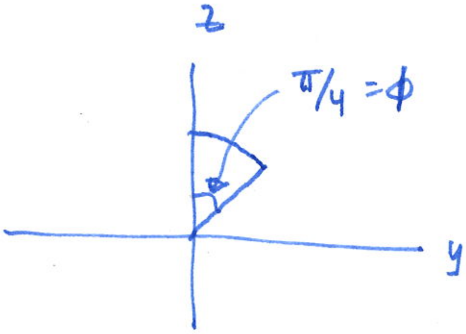
xy shadow:

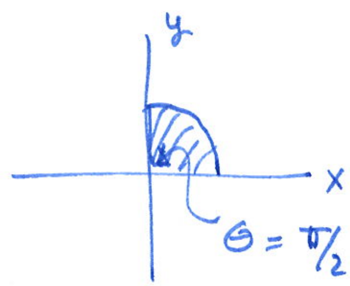




(b) (4pts) Fill in the blanks below in terms of spherical coordinates (No explanations required)

$$I = \int_{0}^{\pi/2} \int_{0}^{\pi/4} \int_{0}^{\sqrt{2}} \underbrace{\rho\cos\theta\sin\phi\,\rho\sin\theta\sin\phi\,\rho^2\sin\phi}_{\rho^4\cos\theta\sin\theta\,\sin^3\phi}\,d\rho\,d\phi\,d\theta$$





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