3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: \[6x^2 y + y^2\]^{1}_{y=-1} = 12x^2; \quad Outer: 4x^3\]^{2}_{0} = 32 .

b) Inner: \[-u \cos t + \frac{1}{2} t^2 \cos u\]^{\pi}_{t=0} = 2u + \frac{1}{2} \pi^2 \cos u
Outer: \[u^2 + \frac{1}{2} \pi^2 \sin u\]^{\pi/2}_{t=0} = \left(\frac{1}{2} \pi\right)^2 + \frac{1}{2} \pi^2 = \frac{3}{4} \pi^2 .

c) Inner: \[\sqrt{x^2 - x^3}\]^{1}_{0} = x^6 - x^3; \quad Outer: \[\frac{1}{4} x^7 - \frac{1}{4} x^4\]^{1}_{0} = \frac{1}{4} - \frac{1}{4} = -\frac{3}{28}.

d) Inner: \[\sqrt{u^2 + 4}\]^{u}_{0} = u\sqrt{u^2 + 4}; \quad Outer: \[\frac{1}{2} (u^2 + 4)^{3/2}\]^{1}_{0} = \frac{1}{2} (5\sqrt{5} - 8).

3A-2

a) (i) \[\int_{-2}^{0} \int_{-x}^{0} dx dy = \int_{0}^{2} \int_{-y}^{y} dx dy\]
(ii) \[\int_{R} dy dx = \int_{0}^{2} \int_{0}^{2a-x^2} dy dx\]

b) i) The ends of \(R\) are at 0 and 2, since \(2x - x^2 = 0\) has 0 and 2 as roots.
\[\int_{R} dy dx = \int_{0}^{2} \int_{0}^{2a-x^2} dy dx\]

ii) We solve \(y = 2x - x^2\) for \(x\) in terms of \(y\); write the equation as \(x^2 - 2x + y = 0\) and solve for \(x\) by the quadratic formula, getting \(x = 1 \pm \sqrt{1-y}\). Note also that the maximum point of the graph is \((1, 1)\) (it lies midway between the two roots 0 and 2). We get
\[\int_{R} dx dy = \int_{0}^{1} \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx dy,\]

b) (i) \[\int_{R} dy dx = \int_{0}^{\sqrt{2}} \int_{0}^{2} dy dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-y^2}} dy dx\]
(ii) \[\int_{R} dx dy = \int_{\sqrt{2}}^{2} \int_{y}^{\sqrt{4-y^2}} dx dy\]

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously \(y^2 = x\) and \(y = x - 2\) (eliminate \(x\)).

The integral \(\int_{R} dy dx\) requires two pieces; \(\int_{R} dx dy\) only one.

3A-3

a) \[\int_{R} x dA = \int_{0}^{2} \int_{0}^{1-x^2/2} x dy dx;\]
Inner: \(x(1 - \frac{1}{2}x)\) \quad Outer: \[\frac{1}{2} x^2 - \frac{1}{8} x^3\]^{2}_{0} = \frac{4}{2} - \frac{8}{8} = \frac{2}{8} .
b) $\int_{R} (2x + y^2) \, dA = \int_{0}^{1} \int_{0}^{1-y^2} (2x + y^2) \, dx \, dy$

Inner: $x^2 + y^2 x_{0}^{1-y^2} = 1 - y^2$; Outer: $y - \frac{1}{3} y^3_{0} = \frac{2}{3}$.

c) $\int_{R} y \, dA = \int_{0}^{1} \int_{y-1}^{1-y} y \, dx \, dy$

Inner: $xy_{y-1}^{1-y} = y[(1-y) - (y-1)] = 2y - 2y^2$ Outer: $y^2 - \frac{2}{3} y^3_{0} = \frac{1}{3}$.

3A-4 a) $\int_{R} \sin^2 x \, dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} \sin^2 x \, dy \, dx$

Inner: $y \sin^2 x_{0}^{\cos x} = \cos x \sin^2 x$ Outer: $\frac{1}{3} \sin^3 x_{-\pi/2}^{\pi/2} = \frac{1}{3} (1 - (-1)) = \frac{2}{3}$.

b) $\int_{R} xy \, dA = \int_{0}^{1} \int_{x^2}^{x} (xy) \, dy \, dx$

Inner: $\frac{1}{2} xy_{x^2}^{x} = \frac{1}{2} (x^3 - x^5)$ Outer: $\frac{1}{2} (x^4 - \frac{x^6}{6})_{0} = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$.

c) The function $x^2 - y^2$ is zero on the lines $y = x$ and $y = -x$, and positive on the region $R$ shown, lying between $x = 0$ and $x = 1$.

Therefore

Volume = $\int_{R} (x^2 - y^2) \, dA = \int_{0}^{1} \int_{-x}^{x} (x^2 - y^2) \, dy \, dx$

Inner: $x^2 y - \frac{1}{3} y^3_{-x}^{x} = \frac{2}{3} x^3$; Outer: $\frac{1}{3} x^4_{0} = \frac{1}{3}$.

3A-5 a) $\int_{0}^{2} \int_{x}^{2} e^{-y^2} \, dy \, dx = \int_{0}^{2} \int_{0}^{y} e^{-y^2} \, dx \, dy = \int_{0}^{2} e^{-y^2} y \, dy = -\frac{1}{2} e^{-y^2}_{0}^{2} = \frac{1}{2} (1 - e^{-4})$

b) $\int_{0}^{1} \int_{u^2}^{\sqrt{1}} \frac{e^u}{u} \, du \, dt = \int_{0}^{1} \int_{0}^{u^2} \frac{e^u}{u} \, du \, dt = \int_{0}^{1} u e^u \, du = (u - 1)e^u_{0}^{1} = 1 - \frac{1}{2} \sqrt{e}$

c) $\int_{0}^{1} \int_{1}^{1+u^4} \frac{1}{1+u^4} \, du \, dx = \int_{0}^{1} \int_{0}^{u^3} \frac{1}{1+u^4} \, dx \, du = \int_{0}^{1} \frac{u^3}{1+u^4} \, du = \frac{1}{4} \ln(1+u^4}_{0}^{1} = \frac{\ln 2}{4}$.

3A-6 0; $2 \int_{S} e^{x} \, dA$, $S$ = right half of $R$; $4 \int_{Q} x^2 \, dA$, $Q$ = first quadrant

0; $4 \int_{Q} x^2 \, dA$; 0

3A-7 a) $x^4 + y^4 \geq 0 \Rightarrow \frac{1}{1 + x^4 + y^4} \leq 1$

b) $\int_{R} \frac{x \, dA}{1 + x^2 + y^2} = \int_{0}^{1} \int_{0}^{1} \frac{x}{1 + x^2} \, dx \, dy = \frac{1}{2} \ln(1 + x^2)_{0}^{1} = \frac{\ln 2}{2} < \frac{1}{2}$.
3B. Double Integrals in polar coordinates

3B-1

a) In polar coordinates, the line \( x = -1 \) becomes \( r \cos \theta = -1 \), or \( r = -\sec \theta \). We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

\[
\int \int_R dr \, d\theta = \int_{\pi/3}^{4\pi/3} \int_{2\pi}^{2\pi/3} dr \, d\theta.
\]

c) We need the polar angle of the intersection points. To find it, we solve the two equations \( r = \frac{3}{2} \) and \( r = 1 - \cos \theta \) simultaneously. Eliminating \( r \), we get \( \frac{3}{2} = 1 - \cos \theta \), from which \( \theta = 2\pi/3 \) and \( 4\pi/3 \). Thus the limits are (no integrand is given):

\[
\int \int_R dr \, d\theta = \int_{2\pi/3}^{4\pi/3} \int_{3/2}^{1-\cos \theta} dr \, d\theta.
\]

d) The circle has polar equation \( r = 2a \cos \theta \). The line \( y = a \) has polar equation \( r \sin \theta = a \), or \( r = a \csc \theta \). Thus the limits are (no integrand):

\[
\int \int_R dr \, d\theta = \int_{\pi/4}^{\pi/2} \int_{2a \cos \theta}^{\pi/2} dr \, d\theta.
\]

3B-2

a) \( \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r \, dr \, d\theta}{r} = \int_0^{\pi/2} \sin 2\theta \, d\theta = \left[ -\frac{1}{2} \cos 2\theta \right]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1. \)

b) \( \int_0^{\pi/2} \int_0^a \frac{r \, dr \, d\theta}{1+r^2} = \frac{\pi}{2} \cdot \frac{1}{2} \ln(1+r^2) \bigg|_0^a = \frac{\pi}{4} \ln(1+a^2). \)

c) \( \int_0^{\pi/4} \int_0^{\sec \theta} \tan^2 \theta \cdot r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta \, d\theta = \frac{1}{6} \tan^3 \theta \bigg|_0^{\pi/4} = \frac{1}{6}. \)

d) \( \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r \, dr \, d\theta}{\sqrt{1-r^2}} \)

Inner: \( -\sqrt{1-r^2} \bigg|_0^{\sin \theta} = 1 - \cos \theta \) Outer: \( \theta - \sin \theta \bigg|_0^{\pi/2} = \pi/2 - 1. \)

3B-3 a) the hemisphere is the graph of \( z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2} \), so we get

\[
\int \int_R \sqrt{a^2-r^2} \, dA = \int_0^{2\pi} \int_0^a \sqrt{a^2-r^2} \, r \, dr \, d\theta = 2\pi \cdot \frac{1}{3} (a^2 - r^2)^{3/2} \bigg|_0^a = 2\pi \cdot \frac{1}{3} a^3 = \frac{2}{3} \pi a^3.
\]
b) \[ \int_0^{\pi/2} \int_0^a (r \cos \theta)(r \sin \theta) r \, dr \, d\theta = \int_0^a r^3 \, dr \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{a^4}{4} \cdot \frac{1}{2} = \frac{a^4}{8}. \]

c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y-axis to compute the volume of just the right side, and double the answer.
\[ \int_{R} \sqrt{x^2 + y^2} \, dA = 2 \int_0^{\pi/2} \int_0^{2\sin \theta} r \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{3} (2 \sin \theta)^3 \, d\theta = 2 \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{32}{9},\] by the integral formula at the beginning of 3B.

d) \[ 2 \int_0^{\pi/2} \int_0^{\sqrt{\cos \theta}} r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{4} \cos^2 \theta \, d\theta = 2 \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{8}. \]

3C. Applications of Double Integration

3C-1 Placing the figure so its legs are on the positive x- and y-axes,

a) M.I. = \[ \int_0^a \int_0^{a-x} x^2 \, dy \, dx \]  
   Inner: \[ x^2 y \Big|_0^{a-x} = x^2(a-x); \]  
   Outer: \[ \frac{1}{3} x^3 a - \frac{1}{4} x^4 \Big|_0^a = \frac{1}{12} a^4. \]

b) \[ \int_{R} (x^2 + y^2) \, dA = \int_{R} x^2 \, dA + \int_{R} y^2 \, dA = \frac{1}{12} a^4 + \frac{1}{12} a^4 = \frac{1}{6} a^4. \]

c) Divide the triangle symmetrically into two smaller triangles, their legs are \( \frac{a}{\sqrt{2}} \);  
Using the result of part (a), M.I. of \( R \) about hypotenuse = \( 2 \cdot \frac{1}{12} \left( \frac{a}{\sqrt{2}} \right)^4 = \frac{a^4}{24} \)

3C-2 In both cases, \( \bar{x} \) is clear by symmetry; we only need \( \bar{y} \).

a) Mass is \[ \int_{R} dA = \int_0^\pi \sin x \, dx = 2 \]
    y-moment is \[ \int_{R} y \, dA = \int_0^\pi \int_0^{\sin x} y \, dy \, dx = \frac{1}{2} \int_0^\pi \sin^2 x \, dx = \frac{\pi}{4}, \] therefore \( \bar{y} = \frac{\pi}{8} \).

b) Mass is \[ \int_{R} y \, dA = \frac{\pi}{4}, \] by part (a).  Using the formulas at the beginning of 3B,
    y-moment is \[ \int_{R} y^2 \, dA = \int_0^\pi \int_0^{\sin x} y^2 \, dy \, dx = 2 \int_0^{\pi/2} \frac{\sin^3 x}{3} \, dx = 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}, \]  
    Therefore \( \bar{y} = \frac{4}{9} \cdot \frac{4}{\pi} = \frac{16}{9\pi} \).
3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the x or y axis. Find the moment of half the segment and double the answer.

(a) (Horizontally, using rectangular coordinates) Note that \(a^2 - c^2 = b^2\).
\[
\int_0^b \int_c^{\sqrt{a^2 - x^2}} y \, dy \, dx = \int_0^b \frac{1}{2} (a^2 - x^2 - c^2) \, dx = \frac{1}{2} \left[ b^3 x - \frac{2}{3} x^3 \right]_0^b = \frac{1}{3} b^3; \quad \text{ans:} \frac{2}{3} b^3.
\]

(b) (Vertically, using polar coordinates). Note that \(x = c\) becomes \(r = c \sec \theta\).

Moment equals \(\int_0^\alpha \int_{c \sec \theta}^{a} (r \cos \theta) r \, dr \, d\theta \quad \text{Inner:} \quad \frac{1}{2} r^3 \cos \theta \int_{c \sec \theta}^{\alpha} = \frac{1}{2} (a^3 \cos \theta - c^3 \sec^2 \theta) \int_{c \sec \theta}^{\alpha} \quad \text{Outer:} \quad \frac{1}{2} \left[ a^3 \sin \theta - c^3 \tan \theta \right]_0^\alpha = \frac{1}{2} (a^2 b - c^2 b) = \frac{1}{3} b^3; \quad \text{ans:} \frac{2}{3} b^3\).

3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive x-axis. By symmetry, the center of mass lies on the x-axis, so we only need find \(\bar{x}\).

Since \(\delta = 1\), the area and mass of the disc are the same: \(\pi a^2 \cdot \frac{2a}{2\pi} = a^2 \alpha\).

\[x\)-moment: \(2 \int_0^\alpha \int_0^a r \cos \theta \cdot r \, dr \, d\theta \quad \text{Inner:} \quad \frac{1}{2} r^3 \cos \theta \int_0^a = \frac{1}{2} a^3 \cos \alpha \int_0^a \quad \text{Outer:} \quad \frac{1}{2} a^3 \sin \alpha \int_0^a = \frac{1}{3} a^3 \sin \alpha \int_0^a \]

\[\bar{x} = \frac{2}{3} a^3 \sin \alpha \int_0^\alpha \sin \alpha \int_0^a = \frac{2}{3} a \cdot \frac{\sin \alpha}{\alpha}.
\]

3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between \(\theta = 0\) and \(\theta = \pi/4\).

\[2 \int_0^{\pi/4} \int_0^{a \sqrt{\cos 2\theta}} r^2 r \, dr \, d\theta = 2 \int_0^{\pi/4} \frac{1}{4} a^4 \cos^2 2\theta \, d\theta \]

Putting \(u = 2\theta\), the above equals \(\frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^4}{16}\).

3D. Changing Variables

3D-1 Let \(u = x - 3y, \ v = 2x + y\). \ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7; \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{7}.
\]
\[
\int \int_R \frac{x - 3y}{2x + y} \, dx \, dy = \frac{1}{7} \int_0^7 \int_1^4 \frac{u}{v} \, dv \, du 
\]

Inner: \(u \ln v \int_1^4 = u \ln 4\); Outer: \(\frac{1}{2} u^2 \ln 4 \right]_1^7 = \frac{49 \ln 4}{2}; \quad \text{Ans:} \frac{1}{7} \frac{49 \ln 4}{2} = 7 \ln 2\).
3D-2 Let \( u = x + y, \ v = x - y \). Then \( \frac{\partial(u,v)}{\partial(x,y)} = 2; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2} \).

To get the \( uv \)-equation of the bottom of the triangular region:
\[ y = 0 \rightarrow u = x, \ v = x \rightarrow u = v. \]
\[
\int_{R} \cos \left( \frac{x-y}{x+y} \right) \ dx \ dy = \frac{1}{2} \int_{0}^{2} \int_{0}^{u} \cos \frac{v}{u} \ dv \ du
\]
Inner: \( u \sin \frac{v}{u} \bigg|_{0}^{u} = u \sin 1 \) \quad Outer: \( \frac{1}{2} u^2 \sin 1 \bigg|_{0}^{2} = 2 \sin 1 \) \quad Ans: \( \sin 1 \)

3D-3 Let \( u = x, \ v = 2y; \ \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \)

Letting \( R \) be the elliptical region whose boundary is \( x^2 + 4y^2 = 16 \) in \( xy \)-coordinates, and \( u^2 + v^2 = 16 \) in \( uv \)-coordinates (a circular disc), we have
\[
\int_{R} (16 - x^2 - 4y^2) \ dy \ dx = \frac{1}{2} \int_{R} (16 - u^2 - v^2) \ dv \ du
\]
\[= \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{4} (16 - r^2) r \ dr \ d\theta = \pi \left( 16 \frac{r^2}{2} - \frac{r^4}{4} \right)_{0}^{4} = 64\pi. \]

3D-4 Let \( u = x + y, \ v = 2x - 3y \); then \( \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5} \).

We next express the boundary of the region \( R \) in \( uv \)-coordinates.
For the \( x \)-axis, we have \( y = 0 \), so \( u = x, \ v = 2x \), giving \( v = 2u \).
For the \( y \)-axis, we have \( x = 0 \), so \( u = y, \ v = -3y \), giving \( v = -3u \).

It is best to integrate first over the lines shown, \( v = c \); this means \( v \) is held constant, i.e., we are integrating first with respect to \( u \). This gives
\[
\int_{R} \int_{1}^{v/2} (2x - 3y) (x + y) \ dy \ dx = \int_{v/3}^{v/2} v^2 u^2 \ du \ dv \frac{5}{5}.
\]
Inner: \( \frac{v^2}{15} \bigg|_{-v/3}^{v/2} = \frac{v^2}{15} \left( \frac{1}{8} - \frac{1}{27} \right) \) \quad Outer: \( \frac{v^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right)_{0}^{4} = \frac{4^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right). \)

3D-5 Let \( u = xy, \ v = y/x \); in the other direction this gives \( y^2 = uv, \ x^2 = u/v \).

We have \( \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x & 1/x \end{vmatrix} = 2y/x = 2v; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v} \), this gives
\[
\int_{R} \int_{1}^{v} (x^2 + y^2) \ dy \ dx = \int_{0}^{3} \int_{1}^{u} \left( \frac{u + uv}{2v} \right) \ dv \ du.
\]
Inner: \( \frac{-u}{2v} + \frac{u v}{2} \bigg|_{1}^{2} = u \left( \frac{-1}{4} + 1 + \frac{1}{2} - \frac{1}{2} \right) = \frac{3u}{4}; \) Outer: \( \frac{3 u^3}{8} \bigg|_{0}^{3} = \frac{27}{8}. \)

3D-8 a) \( y = x^2 \); therefore \( u = x^3, \ v = x \), which gives \( u = v^3 \).
b) We get \( \frac{u}{v} + uv = 1 \), or \( u = \frac{v}{v^2 + 1} \); (cf. 3D-5)