1. For what values of the constant $k$ does the function $f(x, y) = kx^3 + x^2 + 2y^2 - 4x - 4y$ have

   (a) no critical points;
   (b) exactly one critical point;
   (c) exactly two critical points?

   Hint: Consider $k = 0$ and $k \neq 0$ separately.

2. Find and classify all critical points of the following functions.

   (a) $f(x, y) = x^3 - y^3 - 2xy + 6$
   (b) $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
   (c) $f(x, y) = \frac{1}{x^2 + y^2 - 1}$
   (d) $f(x, y) = y \sin x$

3. Suppose $f(x, y)$ satisfies the Laplace’s equation $f_{xx}(x, y) + f_{yy}(x, y) = 0$ for all $x$ and $y$ in $\mathbb{R}^2$. If $f_{xx}(x, y) \neq 0$ for all $x$ and $y$, explain why $f(x, y)$ must not have any local minimum or maximum.

4. Find all absolute maxima and minima of the following functions on the given domains.

   (a) $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate with vertices $(0, 0)$, $(2, 0)$, and $(2, 2)$
   (b) $f(x, y) = x^2 + xy + 3x + 2y + 2$ on the domain $D = \{(x, y) | x^2 \leq y \leq 4\}$
   (c) $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the domain $D = \{(x, y) | x^2 + y^2 \leq 16\}$

5. Use Lagrange multipliers to find the maximum and minimum values of the following functions subject to the given constraint(s).

   (a) $f(x, y) = xy^2$ subject to $x^2 + 2y^2 = 1$
   (b) $f(x, y, z) = xy + z^2$ subject to $y - x = 0$ and $x^2 + y^2 + z^2 = 4$
1. For what values of the constant \( k \) does the function \( f(x, y) = kx^3 + x^2 + 2y^2 - 4x - 4y \) have
   
   (a) no critical points;
   (b) exactly one critical point;
   (c) exactly two critical points?

Hint: Consider \( k = 0 \) and \( k \neq 0 \) separately.

Solution:
Set \( f_x = 0 \) and \( f_y = 0 \) to find critical points:

\[
\frac{df}{dx} = 3kx^2 + 2x - 4 = 0 \quad (1)
\]
\[
\frac{df}{dy} = 4y - 4 = 0 \quad (2)
\]

(2) gives \( y = 1 \). For (1), consider \( k = 0 \) and \( k \neq 0 \) separately.
For \( k = 0 \), (1) becomes \( 2x - 4 = 0 \), or \( x = 2 \). So one critical point at \( (2, 1) \).
For \( k \neq 0 \), use quadratic formula to solve for \( x \).

\[
x = \frac{-2 \pm \sqrt{4 + 48k}}{6k} = \frac{-1 \pm \sqrt{1 + 12k}}{3k}
\]

So critical points are \( \left(\frac{-1 \pm \sqrt{1 + 12k}}{3k}, 1\right) \) if they exist.

Conclusion:

\[
\begin{array}{c}
\text{if } k < -1/12: \text{ no critical points.} \\
\text{if } k = -1/12: \text{ one critical point } (4, 1). \\
\text{if } k > -1/12 \text{ and } k \neq 0: \text{ two critical points } \left(\frac{-1 \pm \sqrt{1 + 12k}}{3k}, 1\right). \\
\text{if } k = 0: \text{ one critical point } (2, 1).
\end{array}
\]

2. Find and classify all critical points of the following functions.

(a) \( f(x, y) = x^3 - y^3 - 2xy + 6 \)

Solution:

Step 1: find critical points

\[
\frac{df}{dx} = 3x^2 - 2y = 0 \quad (1)
\]
\[
\frac{df}{dy} = -3y^2 - 2x = 0 \quad (2)
\]

(1) gives \( y = \frac{3}{2}x^2 \). Substituting into (2) becomes \(-3 \left(\frac{3}{2}x^2\right)^2 - 2x = 0\), or simplified \(-x(27x^3 + 8) = 0\). Hence \( x = 0 \) or \(-2/3\).
If \( x = 0 \), then by (1) \( y = 0 \Rightarrow (0, 0) \)
If \( x = -2/3 \), then by (1) again \( y = 2/3 \Rightarrow (-2/3, 2/3) \).

Hence, critical points at \((0, 0)\) and \((-2/3, 2/3)\).
Step 2: apply second derivative test

\[ f_{xx} = 6x \quad f_{yy} = -6y \quad f_{xy} = -2 \]

At \((0,0)\), \(f_{xx} = 0, f_{yy} = 0, f_{xy} = -2\). So \(D = f_{xx}f_{yy} - (f_{xy})^2 = -4 < 0 \Rightarrow \text{saddle}\)

At \((-2/3, 2/3)\), \(f_{xx} = -4 < 0, f_{yy} = -4, f_{xy} = -2\). So \(D = 12 > 0 \Rightarrow \text{local max}\)

Hence, \text{local max at } (-2/3, 2/3), \text{saddle point at } (0,0)

(b) \(f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8\)

Solution:

Step 1: find critical points

\[ f_x = 3x^2 + 6x = 0 \quad \text{(1)} \]

\[ f_y = 3y^2 - 6y = 0 \quad \text{(2)} \]

We can solve the two equations separately. (1) gives \(x = 0\) and \(-2\). (2) gives \(y = 0\) and \(2\). Hence, there are four critical points at \((0,0)\), \((0,2)\), \((-2,0)\), and \((-2,2)\).

Step 2: apply second derivative test

\[ f_{xx} = 6 + 6 \quad f_{yy} = 6y - 6 \quad f_{xy} = 0 \]

At \((0,0)\), \(f_{xx} = 6, f_{yy} = -6, f_{xy} = 0\), so \(D = -36 < 0 \Rightarrow \text{saddle}\)

At \((0,2)\), \(f_{xx} = 6 > 0, f_{yy} = 6, f_{xy} = 0\), so \(D = 36 > 0 \Rightarrow \text{local min}\)

At \((-2,0)\), \(f_{xx} = -6 < 0, f_{yy} = -6, f_{xy} = 0\), so \(D = 36 > 0 \Rightarrow \text{local max}\)

At \((-2,2)\), \(f_{xx} = -6, f_{yy} = 6, f_{xy} = 0\), so \(D = -36 < 0 \Rightarrow \text{saddle}\)

Hence, \text{local max at } (-2,0), \text{local min at } (0,2), \text{saddle at } (0,0) \text{ and } (-2,2)

(c) \(f(x, y) = \frac{1}{x^2+y^2-1}\)

Solution:

Step 1: find critical points

\[ f_x = -\frac{2x}{(x^2+y^2-1)^2} = 0 \quad \text{(1)} \]

\[ f_y = -\frac{2y}{(x^2+y^2-1)^2} = 0 \quad \text{(2)} \]

(1) gives \(x = 0\) and (2) gives \(y = 0\). The critical point is at \((0,0)\).

Step 2: apply second derivative test

\[ f_{xx} = -\frac{2(x^2+y^2-1)^2 - 2x[2(x^2+y^2-1)(2x)]}{(x^2+y^2-1)^4} \]

\[ f_{yy} = -\frac{2(x^2+y^2-1)^2 - 2y[2(x^2+y^2-1)(2y)]}{(x^2+y^2-1)^4} \]

\[ f_{xy} = \frac{2x(2)(2y)}{(x^2+y^2-1)^3} \]

At \((0,0)\) \(f_{xx} = -2 < 0, f_{yy} = -2, f_{xy} = 0\), So \(D = 4 > 0 \Rightarrow \text{local max}\)

Hence, \text{local max at } (0,0)
(d) \( f(x, y) = y \sin x \)

**Solution:**

**Step 1:** find critical points

\[
\begin{align*}
   f_x &= y \cos x = 0 \\
   f_y &= \sin x = 0
\end{align*}
\]

(2) gives \( x = n\pi \) for all \( n \in \mathbb{Z} \), i.e. integers. Substituting to (1) gives \( \pm y = 0 \), or \( y = 0 \). The critical points are \((n\pi, 0)\) for all \( n \in \mathbb{Z} \).

**Step 2:** apply second derivative test

\[
\begin{align*}
   f_{xx} &= -y \sin x \\
   f_{yy} &= 0 \\
   f_{xy} &= \cos x
\end{align*}
\]

At all \((n\pi, 0)\), \( f_{xx} = 0 \), \( f_{yy} = 0 \), \( f_{xy} = \pm 1 \), so \( D = -1 < 0 \Rightarrow \) saddle

Hence, saddle points at \((n\pi, 0)\) for all \( n \in \mathbb{Z} \).

3. Suppose \( f(x, y) \) satisfies the Laplace’s equation \( f_{xx}(x, y) + f_{yy}(x, y) = 0 \) for all \( x \) and \( y \) in \( \mathbb{R}^2 \). If \( f_{xx}(x, y) \neq 0 \) for all \( x \) and \( y \), explain why \( f(x, y) \) must not have any local minimum or maximum.

**Solution:**

Since the second derivatives exists, the first derivatives must be continuous and \( f(x, y) \) must be differentiable. Also, since there is no boundary on \( \mathbb{R}^2 \), local max/min must occur at critical points.

Suppose there is a critical point, then by second derivative test, \( D = f_{xx}f_{yy} - f_{xy}^2 \). But \( f_{xx} + f_{yy} = 0 \Rightarrow f_{yy} = -f_{xx} \). It follows that \( D = -f_{xx}^2 - f_{xy}^2 < 0 \) when it is given that \( f_{xx} \neq 0 \). Therefore all critical points are saddle points.

4. Find all absolute maxima and minima of the following functions on the given domains.

(a) \( f(x, y) = 2x^2 - 4x + y^2 - 4y + 1 \) on the closed triangular plate with vertices \((0, 0)\), \((2, 0)\), and \((2, 2)\)

**Solution:**

**Step 1:** find interior critical points

\[
\begin{align*}
   f_x &= 4x - 4 = 0 \\
   f_y &= 2y - 4 = 0
\end{align*}
\]

(1) gives \( x = 1 \). (2) gives \( y = 2 \). Critical point at \((1, 2)\), but not in region.

**Step 2:** find boundary critical points and endpoints

Bottom side \( y = 0 \Rightarrow f(x, 0) = 2x^2 - 4x + 1 \).
\[
\frac{df}{dx} = 4x - 4 = 0 \Rightarrow x = 1. \text{ Critical point at } (1, 0)
\]

Right side \( x = 2 \Rightarrow f(2, y) = 8 - 8 + y^2 - 4y + 1 = y^2 - 4y + 1 \).
\[
\frac{df}{dy} = 2y - 4 = 0 \Rightarrow y = 2. \text{ Critical point at } (2, 2).
\]

Hypotenuse \( y = x \Rightarrow f(x, x) = 2x^2 - 4x + x^2 - 4x + 1 = 3x^2 - 8x + 1 \).
\[
\frac{df}{dx} = 6x - 8 = 0 \Rightarrow x = 4/3. \text{ So } y = 4/3. \text{ Critical point at } (4/3, 4/3).
\]

Together with the endpoints of all sides \((0, 0)\), \((2, 0)\), \((2, 2)\).
**Step 3:** compare the values of $f(x, y)$

- $f(1, 0) = -1$
- $f(2, 2) = -3$
- $f(4/3, 4/3) = -13/3 \Leftarrow \text{absolute min}$
- $f(0, 0) = 1 \Leftarrow \text{absolute max}$
- $f(2, 0) = 1 \Leftarrow \text{absolute max}$

Hence, $\text{abs max at } f(2, 0) = f(0, 0) = 1, \text{ abs min at } f(4/3, 4/3) = -13/3$

**(b)** $f(x, y) = x^2 + xy + 3x + 2y + 2$ on the domain $D = \{(x, y) | x^2 \leq y \leq 4\}$

**Solution:**

**Step 1:** find interior critical points

\[
\begin{align*}
    f_x &= 2x + y + 3 = 0 \quad (1) \\
    f_y &= x + 2 = 0 \quad (2)
\end{align*}
\]

(2) gives $x = -2$. Substituting to (1) gives $y = 1$. Critical point at $(-2, 1)$ but not in region.

**Step 2:** find boundary critical points

Top side: $y = 4 \Rightarrow f(x, 4) = x^2 + 4x + 3x + 8 + 2 = x^2 + 7x + 10$

\[
\frac{df}{dx} = 2x + 7 = 0 \Rightarrow x = -7/2 \text{ but not in region}
\]

Parabola: $y = x^2 \Rightarrow f(x, x^2) = x^2 + x^3 + 3x + 2x^2 + 2 = x^3 + 3x^2 + 3x + 2$

\[
\frac{df}{dx} = 3x^2 + 6x + 3 = 3(x + 1)^2 = 0 \Rightarrow x = -1, \text{ then } y = (-1)^2 = 1. \text{ Critical point } (-1, 1).
\]

Together with the endpoints of the two sides $(-2, 4), (2, 4)$.

**Step 3:** compare the values of $f(x, y)$

- $f(-1, 1) = 1$
- $f(-2, 4) = 0 \Leftarrow \text{absolute min}$
- $f(2, 4) = 28 \Leftarrow \text{absolute max}$

Hence, $\text{absolute min at } f(-2, 4) = 0, \text{ absolute max at } f(2, 4) = 28$

**(c)** $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the domain $D = \{(x, y) | x^2 + y^2 \leq 16\}$

**Solution:**

**Step 1:** find interior critical points

\[
\begin{align*}
    f_x &= 4x - 4 = 0 \quad (1) \\
    f_y &= 6y = 0 \quad (2)
\end{align*}
\]

(1) gives $x = 1$. (2) gives $y = 0$. Critical point $(1, 0)$.

**Step 2:** find boundary critical points

Rewrite the boundary $y^2 = 16 - x^2$ or $y = \pm \sqrt{16 - x^2}$, which the endpoints are $(4, 0)$ and $(-4, 0)$.

Then $f$ becomes $f = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43$.

\[
\frac{df}{dx} = -2x - 4 = 0 \Rightarrow x = -2, \text{ } y^2 = 16 - (-2)^2 \Rightarrow y = \pm \sqrt{12}
\]

Critical points at $(-2, \sqrt{12})$ and $(-2, -\sqrt{12})$.

**Step 3:** compare the values of $f(x, y)$

- $f(1, 0) = -7 \Leftarrow \text{absolute min}$
- $f(4, 0) = 11$
5. Use Lagrange multipliers to find the maximum and minimum values of the following functions subject to the given constraint(s).

(a) \( f(x, y) = xy \) subject to \( x^2 + 2y^2 = 1 \)

**Solution:**

**Step 1:** Find critical points on constraint

\[
\begin{align*}
  f(x, y) &= xy, \quad f_x = y, \quad f_y = x \\
  g(x, y) &= x^2 + 2y^2 = 1, \quad g_x = 2x, \quad g_y = 4y
\end{align*}
\]

\[
\begin{align*}
y &= 2\lambda x \\
x &= 4\lambda y \\
x^2 + 2y^2 &= 1
\end{align*}
\]

Substituting (1) into (2) gives \( x = 4\lambda(2\lambda x) \), or \( x(8\lambda^2 - 1) = 0 \) \( \Rightarrow \) \( x = 0 \) or \( \lambda = \pm 1/\sqrt{8} \).

For \( x = 0 \), (2) gives \( y = 0 \), but contradicts with (3). No solution in this case.

For \( \lambda = 1/\sqrt{8} \), (2) gives \( x = \sqrt{2}y \). Substituting into (3) gives \( 2y^2 + 2y^2 = 1 \) \( \Rightarrow \ y = \pm 1/2 \). So \( x = \pm 1/\sqrt{2} \). Critical points at \((1/\sqrt{2}, 1/2), (-1/\sqrt{2}, -1/2)\).

For \( \lambda = -1/\sqrt{8} \), (2) gives \( x = -\sqrt{2}y \). Substituting into (3) gives \( 2y^2 + 2y^2 = 1 \) \( \Rightarrow \ y = \pm 1/2 \). So \( x = \mp 1/\sqrt{2} \). Critical points at \((-1/\sqrt{2}, 1/2), (1/\sqrt{2}, -1/2)\).

**Step 2:** Compare the values of \( f(x, y) \)

\[
\begin{align*}
f(1/\sqrt{2}, 1/2) &= 1/2\sqrt{2} \Leftarrow \text{absolute max} \\
f(-1/\sqrt{2}, -1/2) &= 1/2\sqrt{2} \Leftarrow \text{absolute max} \\
f(-1/\sqrt{2}, 1/2) &= -1/2\sqrt{2} \Leftarrow \text{absolute min} \\
f(1/\sqrt{2}, -1/2) &= -1/2\sqrt{2} \Leftarrow \text{absolute min}
\end{align*}
\]

Hence, \( \text{abs max at } f(1/\sqrt{2}, 1/2) = f(-1/\sqrt{2}, -1/2) = 1/2\sqrt{2} \), \( \text{abs min at } f(-1/\sqrt{2}, 1/2) = f(1/\sqrt{2}, -1/2) = -1/2\sqrt{2} \).

(b) \( f(x, y, z) = xy + z^2 \) subject to \( y - x = 0 \) and \( x^2 + y^2 + z^2 = 4 \)
Solution:

Step 1: Find critical points on constraints

\[ f(x, y) = xy + z^2, \quad f_x = y, \quad f_y = x, \quad f_z = 2z \]
\[ g(x, y) = y - x = 0, \quad g_x = -1, \quad g_y = 1, \quad g_z = 0 \]
\[ h(x, y) = x^2 + y^2 + z^2 = 4, \quad h_x = 2x, \quad h_y = 2y, \quad h_z = 2z \]

\[
\begin{align*}
y &= -\lambda + 2\mu x & (1) \\
x &= \lambda + 2\mu y & (2) \\
2z &= 2\mu z & (3) \\
y - x &= 0 & (4) \\
x^2 + y^2 + z^2 &= 4 & (5)
\end{align*}
\]

(4) gives \( y = x \). Substitute into (1) and (2)
\[
\begin{align*}
x &= -\lambda + 2\mu x & (1a) \\
x &= \lambda + 2\mu x & (2a)
\end{align*}
\]

(1a) – (2a) gives \( \lambda = 0 \). (1) and (2) becomes
\[
\begin{align*}
x &= 2\mu x & (1b) \\
y &= 2\mu y & (2b)
\end{align*}
\]

(1b) and (2b) gives either \( x = y = 0 \) or \( \mu = 1/2 \).

For \( x = y = 0 \), (5) gives \( z = \pm 2 \), and (3) gives \( \mu = 1 \). Critical points at \((0, 0, 2)\) and \((0, 0, -2)\)

For \( \mu = 1/2 \), (3) gives \( z = 0 \). (5) becomes \( x^2 + x^2 = 4 \Rightarrow x = \pm \sqrt{2} \), then \( y = \pm \sqrt{2} \). Critical points at \((\sqrt{2}, \sqrt{2}, 0)\) and \((\sqrt{2}, -\sqrt{2}, 0)\)

Step 2: Compare the values of \( f(x, y) \)
\[
\begin{align*}
f(0, 0, 2) &= 4 \Leftarrow \text{absolute max} \\
f(0, 0, -2) &= 4 \Leftarrow \text{absolute max} \\
f(\sqrt{2}, \sqrt{2}, 0) &= 2 \Leftarrow \text{absolute min} \\
f(-\sqrt{2}, -\sqrt{2}, 0) &= 2 \Leftarrow \text{absolute min}
\end{align*}
\]
Hence, absolute max at \( f(0, 0, 2) = f(0, 0, -2) = 4 \),
absolute min at \( f(\sqrt{2}, \sqrt{2}, 0) = f(-\sqrt{2}, -\sqrt{2}, 0) = 2 \).