Math 200 Problem Set VII

1) Find and classify all the critical points of \( f(x, y) = x^2 + y^2 + x^2 y + 4 \).

2) Find the maximum and minimum values of \( f(x, y) = xy - x^3 y^2 \) when \( (x, y) \) runs over the square \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \).

3) The temperature at all points in the disc \( x^2 + y^2 \leq 1 \) is given by the formula \( T(x, y) = (x + y)e^{-x^2 - y^2} \). Find the maximum and minimum temperatures at points of the disc.

4) Find the high and low points of the surface \( z = \sqrt{x^2 + y^2} \) with \( (x, y) \) varying over the square \( |x| \leq 1, \ |y| \leq 1 \). Discuss the values of \( z_x, z_y \) there. Do not evaluate any derivatives in answering this question.

5) Find the maximum and minimum values of the function \( f(x, y, z) = x + y - z \) on the sphere \( x^2 + y^2 + z^2 = 1 \).

6) Find \( a, b \) and \( c \) so that the volume \( 4\pi abc/3 \) of an ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) passing through the point \( (1, 2, 1) \) is as small as possible.

7) Find the ends of the major and minor axes of the ellipse \( 3x^2 - 2xy + 3y^2 = 4 \).
MATH 200 PROBLEM SET VII SOLUTIONS

1) Find and classify all the critical points of \( f(x, y) = x^2 + y^2 + x^2y + 4 \).

Solution.

\[
\begin{align*}
  f &= x^2 + y^2 + x^2y + 4 \\
  f_x &= 2x + 2xy \\
  f_{xx} &= 2 + 2y \\
  f_{xy} &= 2x \\
  f_y &= 2y + x^2 \\
  f_{yy} &= 2
\end{align*}
\]

The critical points are the solutions of

\[
\begin{align*}
  f_x &= 0 \\
  f_y &= 0
\end{align*}
\]

\[
\iff 
\begin{align*}
  2x(1+y) &= 0 \\
  2y + x^2 &= 0 \\
  x &= 0 \text{ or } y = -1
\end{align*}
\]

When \( x = 0 \), \( y \) must be \(-\frac{1}{2}0^2 = 0\). When \( y = -1 \), \( x^2 \) must be 2. So, there are three critical points: \((0,0), (\pm \sqrt{2}, -1)\).

<table>
<thead>
<tr>
<th>Critical point</th>
<th>( f_{xx}f_{yy} - f_{xy}^2 )</th>
<th>( f_{xx} )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>(2 \times 2 - 0^2 &gt; 0)</td>
<td>2</td>
<td>local min</td>
</tr>
<tr>
<td>((\sqrt{2},-1))</td>
<td>(0 \times 2 - (2\sqrt{2})^2 &lt; 0)</td>
<td>2</td>
<td>saddle pt</td>
</tr>
<tr>
<td>((-\sqrt{2},-1))</td>
<td>(0 \times 2 - (-2\sqrt{2})^2 &lt; 0)</td>
<td>2</td>
<td>saddle pt</td>
</tr>
</tbody>
</table>

2) Find the maximum and minimum values of \( f(x, y) = xy - x^3y^2 \) when \((x, y)\) runs over the square \(0 \leq x \leq 1, 0 \leq y \leq 1\).

Solution.

\[
\begin{align*}
  f(x, y) &= xy - x^3y^2 \\
  f_x(x, y) &= y - 3x^2y^2 \\
  f_y(x, y) &= x - 2x^3y
\end{align*}
\]

First, we find the critical points

\[
\begin{align*}
  f_x &= 0 \iff y(1 - 3x^2y) = 0 \iff y = 0 \text{ or } 3x^2y = 1 \\
  f_y &= 0 \iff x(1 - 2x^2y) = 0 \iff x = 0 \text{ or } 2x^2y = 1
\end{align*}
\]

If \( y = 0 \), we cannot have \( 2x^2y = 1 \), so we must have \( x = 0 \). If \( 3x^2y = 1 \), we cannot have \( x = 0 \), so we must have \( 2x^2y = 1 \). Dividing gives \( 1 = \frac{3x^2y}{2x^2y} = \frac{3}{2} \) which is impossible. So the only critical point in the square is \((0,0)\). There \( f = 0 \).

Next, we look at the part of the boundary with \( x = 0 \). There \( f = 0 \).

Next, we look at the part of the boundary with \( y = 0 \). There \( f = 0 \).

Next, we look at the part of the boundary with \( x = 1 \). There \( f = y - y^2 \). As \( \frac{dy}{dx}(y - y^2) = 1 - 2y \), the max and min of \( y - y^2 \) for \( 0 \leq y \leq 1 \) must occur either at \( y = 0 \), where \( f = 0 \), or at \( y = \frac{1}{2} \), where \( f = \frac{1}{4} \), or at \( y = 1 \), where \( f = 0 \).

Next, we look at the part of the boundary with \( y = 1 \). There \( f = x - x^3 \). As \( \frac{dy}{dx}(x - x^3) = 1 - 3x^2 \), the max and min of \( x - x^3 \) for \( 0 \leq x \leq 1 \) must occur either at \( x = 0 \), where \( f = 0 \), or at \( x = \frac{1}{\sqrt{3}} \), where \( f = \frac{2}{3\sqrt{3}} \), or at \( x = 1 \), where \( f = 0 \).
All together, we have the following candidates for max and min

<table>
<thead>
<tr>
<th>point</th>
<th>(0, 0)</th>
<th>(0, 0 ≤ y ≤ 1)</th>
<th>(0 ≤ x ≤ 1, 0)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
<th>(1/3, 1)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of f</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4/9</td>
<td>0</td>
<td>0</td>
<td>-2/3√3</td>
<td>0</td>
</tr>
</tbody>
</table>

The largest and smallest values of $f$ in this table are $\text{min}=0, \text{max}=\frac{-2}{3\sqrt{3}} \approx 0.385$.

3) The temperature at all points in the disc $x^2 + y^2 ≤ 1$ is given by $T(x, y) = (x + y)e^{-x^2 - y^2}$. Find the maximum and minimum temperatures at points of the disc.

**Solution.**

$$T(x, y) = (x + y)e^{-x^2 - y^2} \quad T_x(x, y) = (1 - 2x^2 - 2xy)e^{-x^2 - y^2} \quad T_y(x, y) = (1 - 2xy - 2y^2)e^{-x^2 - y^2}$$

First, we find the critical points

$$T_x = 0 \iff 2x(x + y) = 1$$

$$T_y = 0 \iff 2y(x + y) = 1$$

As $x + y$ may not vanish, this forces $x = y$ and then $x = y = \pm \frac{1}{2}$. So the only critical points are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

On the boundary $x = \cos t$ and $y = \sin t$, so $T = (\cos t + \sin t)e^{-1}$. This is a periodic function and so takes its max and min at zeroes of $\frac{dT}{dt} = (-\sin t + \cos t)e^{-1}$. That is, when $\sin t = \cos t$, which forces $\sin t = \cos t = \pm \frac{1}{\sqrt{2}}$. All together, we have the following candidates for max and min

<table>
<thead>
<tr>
<th>point</th>
<th>$\left(\frac{1}{2}, \frac{1}{2}\right)$</th>
<th>$\left(-\frac{1}{2}, -\frac{1}{2}\right)$</th>
<th>$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</th>
<th>$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of f</td>
<td>$\frac{1}{\sqrt{e}} \approx 0.61$</td>
<td>$-\frac{1}{\sqrt{e}}$</td>
<td>$\frac{\sqrt{2}}{e} \approx 0.52$</td>
<td>$-\frac{\sqrt{2}}{e}$</td>
</tr>
</tbody>
</table>

The largest and smallest values of $T$ in this table are $\text{min}=-\frac{1}{\sqrt{e}}, \text{max}=-\frac{1}{\sqrt{e}}$.

4) Find the high and low points of the surface $z = \sqrt{x^2 + y^2}$ with $(x, y)$ varying over the square $|x| \leq 1$, $|y| \leq 1$. Discuss the values of $z_x, z_y$ there. Do not evaluate any derivatives in answering this question.

**Solution.** The surface is a cone. The minimum height is at $(0, 0, 0)$. The cone has a point there and the derivatives $z_x$ and $z_y$ do not exist. The maximum height is achieved when $(x, y)$ is as far as possible from $(0, 0)$. The highest points are at $(\pm 1, \pm 1, \sqrt{2})$. There $z_x$ and $z_y$ exist but are not zero. These points would not be the highest points if it were not for the restriction $|x|, |y| \leq 1$.

5) Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 1$.

**Solution.** Define $L(x, y, z, \lambda) = x + y - z - \lambda(x^2 + y^2 + z^2 - 1)$. Then

$$0 = L_x = 1 - 2\lambda x \quad \implies x = \frac{1}{2\lambda}$$

$$0 = L_y = 1 - 2\lambda y \quad \implies y = \frac{1}{2\lambda}$$

$$0 = L_z = -1 - 2\lambda z \quad \implies z = -\frac{1}{2\lambda}$$

$$0 = L_\lambda = x^2 + y^2 + z^2 - 1 \quad \implies 3\left(\frac{1}{2\lambda}\right)^2 - 1 = 0 \quad \implies \lambda = \pm \frac{\sqrt{3}}{2}$$
The critical points are \((-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\), where \(f = -\sqrt{3}\) and \((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})\), where \(f = \sqrt{3}\). So, the max is \(f = \sqrt{3}\) and the min is \(f = -\sqrt{3}\).

6) Find \(a, b\) and \(c\) so that the volume \(4\pi abc/3\) of an ellipsoid \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\) passing through the point \((1, 2, 1)\) is as small as possible.

Solution. Define \(L(a, b, c, \lambda) = \frac{4}{3}\pi abc - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)\). Then

\[
0 = L_a = \frac{4}{3}\pi bc + \frac{2}{a^3} \Rightarrow \frac{2}{a^3}\lambda = -a^3bc \\
0 = L_b = \frac{4}{3}\pi ac + \frac{2}{b^3} \Rightarrow \frac{2}{b^3}\lambda = -\frac{1}{2}ab^3c \\
0 = L_c = \frac{4}{3}\pi ab + \frac{2}{c^3} \Rightarrow \frac{2}{c^3}\lambda = -abc^3 \\
0 = L_\lambda = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} - 1
\]

The equations \(-\frac{3}{2\pi}\lambda = a^3bc = \frac{1}{4}ab^3c\) force \(b = 2a\) (since we want \(a, b, c > 0\)). The equations \(-\frac{3}{2\pi}\lambda = a^3bc = abc^3\) force \(a = c\). Hence

\[
0 = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{a^3} - 1 = \frac{3}{a^3} - 1 \quad \Rightarrow \quad a = c = \sqrt{3}, \quad b = 2\sqrt{3}
\]

7) Find the ends of the major and minor axes of the ellipse \(3x^2 - 2xy + 3y^2 = 4\).

Solution. Let \((x, y)\) be a point on \(3x^2 - 2xy + 3y^2 = 4\). This point is at the end of a major axis when it maximizes its distance from the centre, \((0, 0)\) of the ellipse. It is at the end of a minor axis when it minimizes its distance from \((0, 0)\). So we wish to maximize and minimize \(x^2 + y^2\) subject to \(3x^2 - 2xy + 3y^2 = 4\). Define \(L(x, y, \lambda) = x^2 + y^2 - \lambda(3x^2 - 2xy + 3y^2 - 4)\). Then

\[
0 = L_x = 2x - \lambda(6x - 2y) \quad \Rightarrow \quad (1 - 3\lambda)x + 2\lambda y = 0 \quad (1) \\
0 = L_y = 2y - \lambda(-2x + 6y) \quad \Rightarrow \quad \lambda x + (1 - 3\lambda)y = 0 \quad (2) \\
0 = L_\lambda = 3x^2 - 2xy + 3y^2 - 4
\]

To start, let’s concentrate on the first two equations. Pretend for a couple of minutes, that we already know the value of \(\lambda\) and are trying to find \(x\) and \(y\). The system of equations \((1 + 3\lambda)x - 2\lambda y = 0\), \(-\lambda x + (1 + 3\lambda)y = 0\) has one obvious solution. Namely \(x = y = 0\). But this solution is not acceptable because it does not satisfy the equation of the ellipse. If we have already taken a linear algebra course, you know a system of two linear homogeneous equations in two unknowns have a nonzero solution if and only if the determinant of the matrix of coefficients is zero. (You use this when you find eigenvalues and eigenvectors.) For the equations of interest, this is

\[
\det \begin{bmatrix}
1 - 3\lambda & \lambda \\
\lambda & 1 - 3\lambda
\end{bmatrix} = (1 - 3\lambda)^2 - \lambda^2 = (1 - 2\lambda)(1 - 4\lambda) = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2}, \frac{1}{2}
\]

Even if you have not already taken a linear algebra course, you also come to this conclusion directly when you try to solve the equations. Note that \(\lambda\) cannot be zero because if it is, \((1)\) forces \(x = 0\) and \((2)\) forces \(y = 0\). So equation \((1)\) gives \(y = -\frac{1 - 3\lambda}{2}\) \(x\). Subbing this into equation \((2)\) gives \(\lambda x - \frac{(1 - 3\lambda)^2}{\lambda} x = 0\). To get a nonzero \((x, y)\) we need \(\lambda - \frac{(1 - 3\lambda)^2}{\lambda} = 0 \quad \leftrightarrow \quad \lambda^2 - (1 - 3\lambda)^2 = 0\). By either of these two methods, we now know that \(\lambda\) must be either \(\frac{1}{2}\) or \(\frac{1}{2}\). Subbing these into either \((1)\) or \((2)\) gives

\[
\lambda = \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2}x + \frac{1}{2}y = 0 \quad \Rightarrow \quad x = y \quad \Rightarrow \quad 3x^2 - 2x^2 + 3x^2 = 4 \quad \Rightarrow \quad x = \pm 1 \\
\lambda = \frac{1}{2} \quad \Rightarrow \quad \frac{1}{4}x + \frac{1}{4}y = 0 \quad \Rightarrow \quad x = -y \quad \Rightarrow \quad 3x^2 + 2x^2 + 3x^2 = 4 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}
\]

The ends of the minor axes are \(\pm \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\). The ends of the major axes are \(\pm (1, 1)\).