This Examination paper consists of 7 pages (including this one). Make sure you have all 7.

INSTRUCTIONS:
No memory aids allowed. No calculators allowed. No communication devices allowed.

PLEASE CIRCLE YOUR INSTRUCTOR’S NAME BELOW

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Names of Instructors: Jim Bryan, Dale Peterson, Ian Hewitt, Yariv Dror-Mizrahi, Ed Richmond
Q1  [10 marks]

Find the volume of the region in 3-space which is below the surface \( z = 1 + 3x^2y^2 \) and lies above the region in the xy-plane enclosed by the curves \( x = y^2 \) and \( x = 1 \).

\[
\text{Volume} = \int_1^1 \int_{y^2}^1 (1 + 3x^2y^2) \, dx \, dy
\]

\[
= \int_{y^2}^1 \left[ x + x^3y^2 \right]_y^1 \, dy
\]

\[
= \int_{y^2}^1 \left( 1 + y^2 - y^2 - \frac{1}{9}y^9 \right) \, dy
\]

\[
= \left[ y - \frac{1}{9}y^9 \right]_{y^2}^1
\]

\[
= \left( 1 - \frac{1}{9} \right) - \left( -1 + \frac{1}{9} \right) = 2 - \frac{2}{9} = \frac{16}{9}
\]
Q2 [12 marks]

Suppose \( T(x, y, z) = xy^2 - x + x^2z + yz^2 \) gives the temperature at the point \((x, y, z)\) in space.

(a) Find an equation of the plane tangent at \((1, 2, 1)\) to the level surface of \( T \) passing through that point. [4pts]

\[
\nabla T = \langle y^2 - 1 + 2xz, 2xy + z^2, x^2 + 2yz \rangle
\]

\[
\nabla T(1, 2, 1) = \langle 4 - 1 + 2, 4 + 1, 1 + 4 \rangle = \langle 5, 5, 5 \rangle
\]

Equation of tangent plane: \( 5(x - 1) + 5(y - 2) + 5(z - 1) = 0 \)

or \( (x - 1) + (y - 2) + (z - 1) = 0 \)

or \( x + y + z = 4 \)

(b) At time \( t = 0 \), a fly passes through \((1, 2, 1)\) moving toward the point \((4, 2, 5)\) at speed of 1 unit/sec. Calculate \( \frac{dT}{dt} \) at \( t = 0 \) for the fly. [4pts]

Direction of fly: \( \langle 4, 2, 5 \rangle - \langle 1, 2, 1 \rangle = \langle 3, 0, 4 \rangle \)

Unit vector \( \vec{u} = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle \)

\[
\frac{dT}{dt} \bigg|_{t=0} = \nabla T \cdot \vec{u} = \langle 5, 5, 5 \rangle \cdot \langle \frac{3}{5}, 0, \frac{4}{5} \rangle = 7
\]
(c) A worm crawling on the plane \(2x - y + 2z = 2\) passes through the point \((1, 2, 1)\). The worm wishes to keep his temperature constant while increasing \(z\). In which direction should the worm move? Express your answer as a unit vector. [4pts]

Direction of worm is perpendicular to \(\langle 2, -1, 2 \rangle\)
since worm crawls on the plane \(2x - y + 2z = 2\)

Direction of worm is perpendicular to \(\langle 5, 5, 5 \rangle\) since worm wants his temperature constant.

Thus we seek a unit vector, with positive \(\hat{k}\) component, which is perpendicular to \(\langle 2, -1, 2 \rangle \& \langle 5, 5, 5 \rangle\)

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & -1 & 2 \\
5 & 5 & 5 \\
\end{vmatrix} = 5 \\
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & -1 & 2 \\
1 & 1 & 1 \\
\end{vmatrix} = 5 \langle -3, 0, 3 \rangle
\]

Make it a unit vector \(\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle\)
Q3 [12 marks]

Consider the following iterated integral

\[ \int_{0}^{2} \int_{y = 2 - \sqrt{4-x^2}}^{x} f(x, y) \, dy \, dx. \]

(a) Sketch the region of integration. Be sure to label your axes and clearly mark x and y values on the axes. Give the coordinates of any intersection points. [4pt]

\[ y = 2 - \sqrt{4-x^2} \]
\[ \Rightarrow (y-2)^2 = 4-x^2 \]
\[ \Rightarrow (y-2)^2 + x^2 = 2^2 \quad \text{circle of radius 2 centered at (0,2)} \]
\[ \Rightarrow x^2 = 4 - (y-2)^2 \]
\[ x = \sqrt{4-(y-2)^2} \]

(b) Change the order of integration to \( dx \, dy \). [4pt]

\[ \int_{y=0}^{2} \int_{x=y}^{\sqrt{4-(y-2)^2}} f(x, y) \, dx \, dy \]

(c) Convert the integral to polar coordinates. [4pt]

\[ \int_{\theta=0}^{\pi/4} \int_{r=0}^{4 \sin \theta} f(r \cos \theta, r \sin \theta) \, rdr \, d\theta \]

\[ y^2 = 4y + y + x^2 = y \]
\[ y^2 + x^2 = 4y \]
\[ r^2 = 4r \sin \theta \]
\[ r = 4 \sin \theta \]
Q4 [16 marks]

Consider the function \( f(x, y) = (4y + 7)e^{-x^2-y^2} \) on the domain \( x^2 + y^2 \leq 1 \).

(a) Find all critical points of \( f \) which are inside the domain [4 pts]

\[
\begin{align*}
    f_x &= -2x(4y+7)e^{-x^2-y^2} \\
    f_y &= 4e^{-x^2-y^2} - 2y(4y+7)e^{-x^2-y^2}
\end{align*}
\]

\( f_x = 0 \Rightarrow -2x(4y+7) = 0 \quad f_y = 0 \Rightarrow 4 - 2y(4y+7) = 0 \)

so \( x = 0 \) or \( (4y+7) = 0 \)

so \( x = 0 \) and \( 4 - 2y(4y+7) = 0 \) \( \Rightarrow \) \( 4 - 8y^2 - 14y = 0 \) \( \Rightarrow \) \( 8y^2 + 14y - 4 = 0 \)

\( \Rightarrow (8y - 2)(y + 2) = 0 \) \( \Rightarrow y = -2 \) or \( \frac{1}{4} \)

Critical points are \( (0, -2) \) and \( (0, \frac{1}{4}) \) only \( (0, \frac{1}{4}) \) is in the domain

(b) Classify each of the critical points on the inside of the domain as a “local maximum”, “local minimum”, “saddle points”, or “discriminant is zero”. [4 pts]

\[
\begin{align*}
    f_{xx} &= -2(4y+7)e^{-x^2-y^2} + 4x^2(4y+7)e^{-x^2-y^2} \\
    f_{xy} &= -8ye^{-x^2-y^2} \\
    f_{yy} &= (4 - 2y(4y+7))e^{-x^2-y^2} \\
    f_{yx} &= f_{xy} = (4 - 8y^2 - 14y)e^{-x^2-y^2} \\
    f_x(0, \frac{1}{4}) &= -2(1+7)e^{-\frac{1}{16}} = -16e^{-\frac{1}{16}} \\
    f_y(0, \frac{1}{4}) &= (4 - 14)e^{-\frac{1}{16}} - \frac{2}{4} \left( 4 - \frac{1}{2} - \frac{7}{2} \right)e^{-\frac{1}{16}} = -18e^{-\frac{1}{16}} \\
    f_{xx}(0, \frac{1}{4}) &= 0 \quad \text{so} \quad D = f_{xx}f_{yy} - f_{xy}^2 = (-16)e^{-\frac{1}{16}} (-18)e^{-\frac{1}{16}} > 0 \\
    \quad \text{and} \quad f_{xx} < 0 \quad \Rightarrow \quad (0, \frac{1}{4}) \quad \text{is a local maximum} 
\end{align*}
\]
(c) Use the method of Lagrange multipliers to find the maximum and minimum values of \( f \) on the boundary of the domain. [6pt]

\[
1. \quad g(x,y) = x^2 + y^2 = 1 \quad f_x = \lambda g_x \quad f_y = \lambda g_y
\]

\[
2. \quad -2x(4y+7)e^{-x^2-y^2} = 2x \quad \Rightarrow \quad x = 0 \quad \text{or} \quad \lambda = -(4y+7)e^{-x^2-y^2} 
\]

\[
3. \quad (4-8y^2-14y)e^{-x^2-y^2} = \lambda \, 2y
\]

\[
\lambda \, 2y = (4-8y^2-14y)e^{-x^2-y^2} 
\]

\[
\lambda = \frac{(4-8y^2-14y)e^{-x^2-y^2}}{2y} 
\]

\[
y-8y^2-14y = -8y^2-14y \quad \Rightarrow \quad y = 0 
\]

\[
\text{no solution}
\]

\[
\text{x=0 sub into 1} \Rightarrow y = \pm 1
\]

\[
\text{so (x,y) = (1,0), (-1,0) are solutions}
\]

\[
\text{so (0,1) and (0,-1) are the only possibilities}
\]

\[
f(0,1) = 11e^{-1} \quad f(0,-1) = 3e^{-1}
\]

\[
\text{max on boundary} \quad \text{min on boundary}
\]

(d) Find the absolute maximum and minimum values of \( f \) on its whole domain. You may use the fact that \( e^{15/16} > 11/8 \). [2 pts]

Abs. max/min must occur at a crit pt on the interior or a max/min on the boundary. So only possibilities are (1,0), (0,1), (0,1/4)

\[
f(0,1) = 11e^{-1} \quad \text{since } e^{15/16} > 11/8
\]

\[
f(0,-1) = 3e^{-1} \quad \Rightarrow \quad 8e^{-1/16} > 11
\]

\[
f(0,1/4) = 8e^{-1/16} \quad \Rightarrow \quad 8e^{-1/16} > 11e^{-1}
\]

\[
\text{so } 3e^{-1} \text{ is absolute min and } 8e^{-1/16} \text{ is absolute max.}
\]