Midterm 1 — May 31st, 2017 Duration: 50 minutes

This test has 4 ***questions*** *on* ***8 pages****, for a total of 25 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations. Write legibly and in a coherent order.
- Continue on the back of the previous page if you run out of space or use the blank page at the end. If you continue a problem on a different page, indicate this clearly at the bottom of the problem's original page.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: Solutions Last Name: ____________

Student-No: ____________ Class (253 or 200): ____________

Signature: ____________

Question:	1	2	3	4	Total
Points:	9	4	6	6	25
Score:					

3 marks 1. (a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ where z is implicitly given as a function of x and y by the equation $xyz = e^{x+y+z}$.

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}\left(e^{x+y+z}\right)$$

$$yz + xy\frac{\partial z}{\partial x} = e^{x+y+z}\frac{\partial}{\partial x}(x+y+z) = \left(1+\frac{\partial z}{\partial x}\right)e^{x+y+z}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}\left(xy - e^{x+y+z}\right) = e^{x+y+z} - yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{e^{x+y+z} - yz}{xy - e^{x+y+z}}}$$

3 marks (b) Let $z = e^{xy} + y$ and suppose that $x = f(t)$ and $y = g(t)$ where f and g are functions such that $f(0) = 2$, $f'(0) = -1$, $g(0) = 3$, and $g'(0) = 2$. Compute $\frac{dz}{dt}$ when $t = 0$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}\frac{dx}{dt} + (xe^{xy}+1)\frac{dy}{dt}$$

at $t=0$ $\quad x = f(0) = 2 \quad y = g(0) = 3 \quad \frac{dx}{dt} = f'(0) = -1 \quad \frac{dy}{dt} = g'(0) = 2$

so

$$\left.\frac{dz}{dt}\right|_{t=0} = 3e^6(-1) + (2e^6+1)2$$

$$= \boxed{2 + e^6}$$

3 marks (c) Let $z = \frac{1}{2}\log(x^2+y^2)$ (the log here is the natural log). Which of the following partial differential equations does z satisfy? Clearly circle all which apply:

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0} \qquad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0} \qquad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{1} \qquad \frac{\partial z}{\partial y} = \underbrace{\frac{y}{x^2+y^2}}_{2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \underbrace{\frac{y^2-x^2}{(x^2+y^2)^2}}_{3}$$

and then by symmetry $(x \leftrightarrow y)$

$$\frac{\partial^2 z}{\partial y^2} = \underbrace{\frac{x^2-y^2}{(x^2+y^2)^2}}_{} \; 4$$

$$\frac{\partial^2 z}{\partial y \partial \cancel{x}} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \underbrace{\frac{-2yx}{(x^2+y^2)^2}}_{5}$$

3 + 4 = 0 $\qquad$ 3 − 4 ≠ 0

5 + 2 1 2 = 0 $\qquad$ $1^2 + 2^2 \neq 1$

2. Label each equation by the corresponding plot.

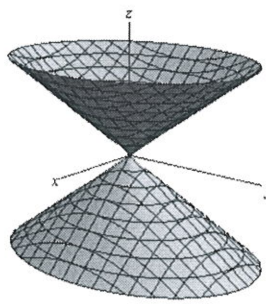
marks (a) $z^2 = x^2 + y^2$. $\boxed{A}$

marks (b) $z = x^2 - y^2$. $\boxed{E}$

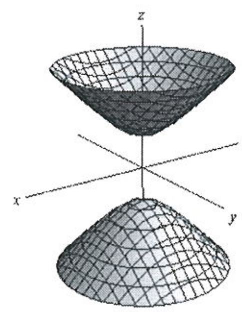
marks (c) $z^2 = x^2 + y^2 - 1$. $\boxed{F}$

marks (d) $z^2 = x^2 + y^2 + 1$. $\boxed{B}$

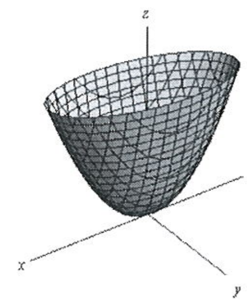
think about the trace curves given by $x=0$ and $y=0$ and the contour curves given by $z=0$ (and $z=C$ for $C\neq 0$)



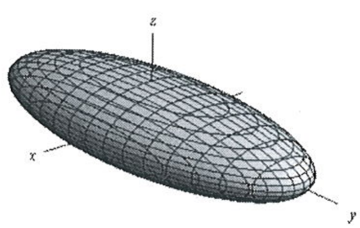
A



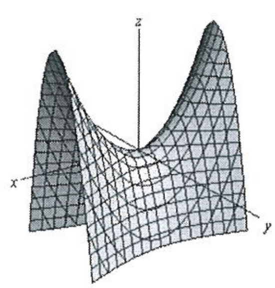
B



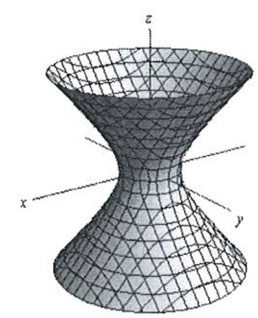
C



D



E



F

3. Let $f(x,y) = \sqrt{xy}$.

3 marks

(a) Find the linear approximation of $f(x,y)$ near $(x_0, y_0) = (40, 10)$.

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f_x = \frac{1}{2}\frac{y}{\sqrt{xy}} \qquad f_y = \frac{1}{2}\frac{x}{\sqrt{xy}} \qquad f(40,10) = 20,\ f_x(40,10) = \frac{1}{2}\frac{10}{20} = \frac{1}{4}$$

$$f_y(40,10) = \frac{1}{2}\frac{40}{20} = 1$$

$$f(x,y) \approx 20 + \frac{1}{4}(x-40) + (y-10)$$

2 marks
2

(b) Use your answer in part (a) to find a good estimate of $\sqrt{39}$ as a rational number (i.e. a fraction). Hint: 9 is close to 10 and is a perfect square.

$$f(39,9) = \sqrt{39 \cdot 9} = 3\sqrt{39}$$

$$f(39,9) \approx 20 + \frac{1}{4}(39-40) + (9-10) = 20 - \frac{1}{4} - 1 = 18\frac{3}{4}$$

so $3\sqrt{39} \approx 18\frac{3}{4} \Rightarrow \boxed{\sqrt{39} \approx 6\frac{1}{4}}$

1 mark

(c) Write your estimate as a decimal number and compare it to the actual value $\sqrt{39} = 6.2449\cdots$. How close is your estimate?

$$6\frac{1}{4} = 6.25$$

$$6.25 - 6.2449\ldots = 0.0051\ldots$$

we are off by less than 1%.

4. Consider the triangle given by the three points $p_1 = (2, 1, 2)$, $p_2 = (-2, 2, 1)$, and $p_3 = (-1, -2, 2)$.

2 marks (a) Find the length of the edge from p_1 to p_2 and the edge from p_1 to p_3.

vector from p_1 to p_2 is $\langle -2,2,1\rangle - \langle 2,1,2\rangle = \langle -4,1,-1\rangle$

" " " " p_3 " $\langle -1,-2,2\rangle - \langle 2,1,2\rangle = \langle -3,-3,0\rangle$

length of edge $\overline{p_1p_2} = |\langle -4,1,-1\rangle| = \sqrt{16+1+1} = \sqrt{18} = 3\sqrt{2}$

" " " $\overline{p_1p_3} = |\langle -3,-3,0\rangle| = \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2}$

$\boxed{\text{both edges have length } 3\sqrt{2}}$

1 mark (b) Find the angle between those two edges.

$\langle -4,1,-1\rangle \cdot \langle -3,-3,0\rangle = (3\sqrt{2})(3\sqrt{2})\cos\theta$

"

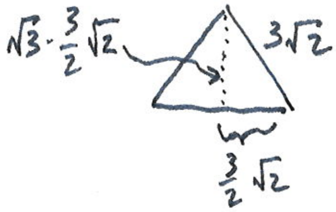
$12 - 3 = 9$

so $9 = 9\cdot 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \boxed{\theta = \frac{\pi}{3}}$

1 mark (c) Find the area of the triangle.

method 1: since side lengths of 2 edges are equal with an angle of $\pi/3$, the triangle is equilateral:



$$\text{Area} = \frac{1}{2}\,\text{base}\cdot\text{height} = \frac{1}{2}(3\sqrt{2})\left(\sqrt{3}\cdot\frac{3}{2}\sqrt{2}\right) = \boxed{\frac{9}{2}\sqrt{3}}$$

method 2: $|\langle -4,1,-1\rangle \times \langle -3,-3,0\rangle| =$ Area of parallelogram $= 2$ Area of triangle

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ -3 & -3 & 0 \end{vmatrix} = |\langle -3, 3, 15\rangle| = 3|\langle -1,1,5\rangle| = 3\sqrt{1+1+25} = 3\sqrt{27} = 9\sqrt{3}$$

so area of triangle is $\boxed{\frac{9}{2}\sqrt{3}}$

2 marks (d) Find the equation of the plane containing the triangle.

$\langle -4,1,-1\rangle \times \langle -3,-3,0\rangle = \langle -3,3,15\rangle$ is normal to the plane and $(-1,-2,2)$ is contained in the plane. So the equation of the plane is

$$-3(x+1) + 3(y+2) + 15(z-2) = 0$$

$$\iff \quad -(x+1) + (y+2) + 5(z-2) = 0$$

$$\iff \quad -x - 1 + y + 2 + 5z - 10 = 0$$

$$\iff \quad \boxed{-x + y + 5z = 9}$$