1. Prove that the following differential equations are satisfied by the given functions:

(a) \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \), where \( u = (x^2 + y^2 + z^2)^{-1/2} \).

(b) \( x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = -2w \), where \( w = \left( x^2 + y^2 + z^2 \right)^{-1} \).

2. Show that the function \( u = t^{-1}e^{-(x^2+y^2)/4t} \) satisfies the two dimensional heat equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \).

3. (a) Find an equation of the tangent plane to the surface \( x^2 + y^2 + z^2 = 9 \) at the point \( (2, 2, 1) \).

(b) At what points \( (x, y, z) \) on the surface in part (a) are the tangent planes parallel to \( 2x + 2y + z = 1 \)?

4. Find the points on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \) where the tangent plane is parallel to the plane \( 3x - y + 3z = 1 \).

5. (a) Find an equation for the tangent line to the curve of intersection of the surfaces \( x^2 + y^2 + z^2 = 9 \) and \( 4x^2 + 4y^2 - 5z^2 = 0 \) at the point \( (1, 2, 2) \).

(b) Find the radius of the sphere whose center is \( (-1, -1, 0) \) and which is tangent to the plane \( x + y + z = 1 \).

6. Find the point(s) on the surface \( z = xy \) that are nearest to the point \( (0, 0, 2) \).

7. Let \( f(x, y, z) \) be the function defined by \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \). Determine an equation for the normal line of the surface \( f(x, y, z) = 3 \) at the point \( (-1, 2, 2) \).

8. Let \( f(x, y, z) = \frac{xy}{z} \). Measurements are made and it is found that \( x = 10, y = 10, z = 2 \).
   If the maximum error made in each measurement is 1% find the approximate percentage error made in computing the value of \( f(10, 10, 2) \).

9. Find all points on the surface given by
   \[ (x - y)^2 + (x + y)^2 + 3z^2 = 1 \]
   where the tangent plane is perpendicular to the plane \( 2x - 2y = 13 \).

10. Find all points at which the direction of fastest change of \( f(x, y) = x^2 + y^2 - 2x - 4y \) is \( \vec{i} + \vec{j} \).
11. The surface \( x^4 + y^4 + z^4 + xyz = 17 \) passes through \((0, 1, 2)\), and near this point the surface determines \( x \) as a function, \( x = F(y, z) \), of \( y \) and \( z \).

(a) Find \( F_y \) and \( F_z \) at \((x, y, z) = (0, 1, 2)\).

(b) Use the tangent plane approximation (otherwise known as linear, first order or differential approximation) to find the approximate value of \( x \) (near 0) such that \((x, 1.01, 1.98)\) lies on the surface.

12. Let \( f(x, y) \) be a differentiable function, and let \( u = x + y \) and \( v = x - y \). Find a constant \( \alpha \) such that

\[
(f_x)^2 + (f_y)^2 = \alpha((f_u)^2 + (f_v)^2).
\]

13. Find the directional derivative \( D_\vec{u}f \) at the given point in the direction indicated by the angle

(a) \( f(x, y) = \sqrt{3x - 4y}, \ (2, 1), \ \theta = -\pi/6. \)

(b) \( f(x, y) = x \sin(xy), \ (2, 0), \ \theta = \pi/3. \)

14. Compute the directional derivatives \( D_\vec{u}f \), where:

(a) \( f(x, y) = \ln(x^2 + y^2), \ \vec{u} \) is the unit vector pointing from \((0, 0)\) to \((1, 2)\).

(b) \( f(x, y, z) = \frac{1}{\sqrt{x^2 + 2y^2 + 3z^2}}, \ \vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle. \)

15. Find all points \((x, y, z)\) such that \( D_\vec{u}f(x, y, z) = 0 \), where \( \vec{u} = \langle a, b, c \rangle \) is a unit vector and \( f(x, y, z) = \sqrt{ax^2 + by^2 + cz^2}. \)

16. Compute the cosine of the angle between the gradient \( \nabla f \) and the positive direction of the \( z \)-axis, where \( f(x, y, z) = x^2 + y^2 + z^2. \)

17. The temperature at a point \((x, y, z)\) is given by \( T(x, y, z) = 200e^{-x^2 - 3y^2 - 9z^2}. \)

(a) Find the rate of change of temperature at the point \( P(2, -1, 2) \) in the direction towards the point \((3, -3, 3)\).

(b) In which direction does the temperature increase the fastest at \( P \)?

(c) Find the maximum rate of increase at \( P \).
1. Prove that the following differential equations are satisfied by the given functions:

(a) \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \), where \( u = (x^2 + y^2 + z^2)^{-1/2} \).

(b) \( x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = -2w \), where \( w = (x^2 + y^2 + z^2)^{-1} \).

Solution:

(a) \( \frac{\partial u}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2} \) and \( \frac{\partial^2 u}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2} \).

By symmetry we see that

\[ \frac{\partial^2 u}{\partial y^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2} \]

\[ \frac{\partial^2 u}{\partial z^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2} \]

Adding up clearly gives 0.

(b) \( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = -2x^2(x^2 + y^2 + z^2)^{-2} - 2y^2(x^2 + y^2 + z^2)^{-2} - 2z^2(x^2 + y^2 + z^2)^{-2} = -2w \).

2. Show that the function \( u = t^{-1}e^{-(x^2+y^2)/4t} \) satisfies the two dimensional heat equation

\( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \).

Solution:

\[ \frac{\partial u}{\partial t} = -t^{-2}e^{-(x^2+y^2)/4t} + \frac{x^2 + y^2}{4t^3}e^{-(x^2+y^2)/4t} \]

\[ \frac{\partial u}{\partial x} = -\frac{x}{2t^2}e^{-(x^2+y^2)/4t} \frac{\partial u}{\partial y} = -\frac{y}{2t^2}e^{-(x^2+y^2)/4t} \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{x^2}{4t^3}e^{-(x^2+y^2)/4t} \]

\[ \frac{\partial^2 u}{\partial y^2} = -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{y^2}{4t^3}e^{-(x^2+y^2)/4t} \]

It is now clear that \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \).

3. (a) Find an equation of the tangent plane to the surface \( x^2 + y^2 + z^2 = 9 \) at the point \((2, 2, 1)\).
4. Find the points on the ellipsoid \( x^2 + 5y^2 + 2z^2 = 1 \) for which the normal \( n \) is parallel to the normal to the plane \( x + 2y + z = 0 \).

Solution:

\( f(x, y, z) = x^2 + 5y^2 + 2z^2 \) then a normal of the surface \( x^2 + 5y^2 + 2z^2 = 1 \) at \( (2, 2, 1) \)

\[ n = \nabla f(x, y, z)|_{(2,2,1)} = (2x i + 2y j + 2z k)|_{x=2,y=2,z=1} = 4i + 4j + 2k. \]

From part (a) we see that one of the points is \( (2, 2, 1) \).

(b) By taking gradients (up to constant multiples) we see that the respective normals \( \mathbf{n} \) are \( \mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \) and \( \mathbf{n}_2 = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \). Thus, \( \mathbf{n}_1 \times \mathbf{n}_2 =\) a direction vector at \( (1, 2, 2) \) for the curve of intersection is \( \mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = -18\mathbf{i} + 9\mathbf{j}. \)

5. (a) Find an equation for the tangent line to the curve of intersection of the surfaces

\[ x^2 + y^2 + z^2 = 1 \quad \text{and} \quad 4x^2 + 4y^2 - 5z^2 = 0 \]

at \( (1, 2, 2) \).

(b) Find the radius of the sphere whose center is \( (-1, -1, 0) \) and which is tangent to the plane \( x + y + z = 1 \).

Solution:

(a) By taking gradients (up to constant multiples) we see that the respective normals at \( (1, 2, 2) \) are \( \mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \) and \( \mathbf{n}_2 = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \). Thus, a direction vector at \( (1, 2, 2) \) for the curve of intersection is \( \mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = -18\mathbf{i} + 9\mathbf{j}. \) Removing a factor of 9 we see that a direction vector is \( \mathbf{v} = -2\mathbf{i} + \mathbf{j}, \) and therefore the equation of the tangent line is \( x = 1 - 2t, y = 2 + t, z = 2. \)

(b) The sphere will have the equation \( (x + 1)^2 + (y + 1)^2 + z^2 = r^2 \) for some \( r \). In order for this sphere to be tangent to the plane \( x + y + z = 1 \) it is necessary
that the “radius vector” be proportional to the normal vector to the plane, that is
\((x + 1, y + 1, z) = \lambda(1, 1, 1)\) for some \(\lambda\). But we must also have \(x + y + z = 1\) and
therefore \(\lambda = 1\). It follows that \(r = \sqrt{3}\).

6. Find the point(s) on the surface \(z = xy\) that are nearest to the point \((0, 0, 2)\).

Solution:
Let \(F(x, y, z) = z - xy\). Thus the surface is the level surface \(F(x, y, z) = 0\). We want
to find all points \((x, y, z)\) on the surface where the gradient \(\nabla F(x, y, z)\) is parallel to
the vector pointing from \((0, 0, 2)\) to \((x, y, z)\). Therefore
\[x = -\lambda y, y = -\lambda x, z - 2 = \lambda, \text{ and } z = xy.\]
The solutions are
\[(x, y, z) = (0, 0, 0), \lambda = -2; (x, y, z) = (1, 1, 1), \lambda = -1; (x, y, z) = (-1, -1, 1), \lambda = -1.\]
Clearly \(\exists\) closest point(s). They are \((1, 1, 1)\) and \((-1, -1, 1)\).

7. Let \(f(x, y, z)\) be the function defined by \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\). Determine an
equation for the normal line of the surface \(f(x, y, z) = 3\) at the point \((-1, 2, 2)\).

Solution:
A normal to the surface \(f(x, y, z) = 3\) at \((-1, 2, 2)\) is \(\vec{n} = \langle -1/3, 2/3, 2/3 \rangle\). Thus an
equation for the normal line is
\[x = -1 - \frac{1}{3}t, y = 2 + \frac{2}{3}t, z = 2 + \frac{2}{3}t, -\infty < t < \infty.\]

8. Let \(f(x, y, z) = \frac{xy}{z}\). Measurements are made and it is found that \(x = 10, y = 10, z = 2\).
If the maximum error made in each measurement is 1% find the approximate percentage
error made in computing the value of \(f(10, 10, 2)\).

Solution: The calculated value is \(f(10, 10, 2) = 50\), with errors
\[-0.1 \leq \Delta x \leq 0.1, \ -0.1 \leq \Delta y \leq 0.1 \text{ and } -0.02 \leq \Delta z \leq 0.02.\]
The approximate error made is
\[f(10 + \Delta x, 10 + \Delta y, 2 + \Delta z) - f(10, 10, 2) \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta x\]
\[= 5\Delta x + 5\Delta y - 25\Delta z.\]
Then we have \(-1.5 \leq 5\Delta x + 5\Delta y - 25\Delta z \leq 1.5\), and so the approximate percentage
error is \(\frac{1.5}{50} \times 100\% = 3\%.\)
9. Find all points on the surface given by

$$(x - y)^2 + (x + y)^2 + 3z^2 = 1$$

where the tangent plane is perpendicular to the plane $2x - 2y = 13$.

Solution:
A normal to the surface $(x - y)^2 + (x + y)^2 + 3z^2 = 1$ is $\vec{n} = <4x, 4y, 6z>$. Thus we want to solve simultaneously the equations $(x - y)^2 + (x + y)^2 + 3z^2 = 1$ and $<4x, 4y, 6z> \cdot <2, -2, 0> = 0$. Thus the points are $(x, x, z)$, where $x, z$ lie on the ellipse $4x^2 + 3z^2 = 1$.

10. Find all points at which the direction of fastest change of $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\vec{i} + \vec{j}$.

Solution:
The direction of fastest change is in the direction of $\nabla f = <2x - 2, 2y - 4>$. Therefore we want $\nabla f = <2x - 2, 2y - 4> = (\lambda, \lambda)$, that is $2x - 2 = 2y - 4$. Therefore the points are $(x, y) = (x, x + 1), -\infty < x < \infty$.

11. The surface $x^4 + y^4 + z^4 + xyz = 17$ passes through $(0, 1, 2)$, and near this point the surface determines $x$ as a function, $x = F(y, z)$, of $y$ and $z$.

(a) Find $F_y$ and $F_z$ at $(x, y, z) = (0, 1, 2)$.

(b) Use the tangent plane approximation (otherwise known as linear, first order or differential approximation) to find the approximate value of $x$ (near 0) such that $(x, 1.01, 1.98)$ lies on the surface.

Solution:
(a) To find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ at $(y, z) = (1, 2)$ we differentiate the equation

$$x^4 + y^4 + z^4 + xyz = 17$$

with respect to $y, z$;

then put $x = 0, y = 1, z = 2$, and finally solve for $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$:

$$\frac{\partial}{\partial y}(x^4 + y^4 + z^4 + xyz) = 0 \implies 4x^3\frac{\partial x}{\partial y} + 4y^3 + \frac{\partial x}{\partial y}yz + xz = 0 \implies \frac{\partial x}{\partial y} = -2$$

$$\frac{\partial}{\partial z}(x^4 + y^4 + z^4 + xyz) = 0 \implies 4x^3\frac{\partial x}{\partial z} + 4z^3 + \frac{\partial x}{\partial z}yz + xy = 0 \implies \frac{\partial x}{\partial z} = -16$$

(b) The tangent plane approximation is

$$F(y + \Delta y, z + \Delta z) \approx F(y, z) + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z.$$ 

In this case $F(1, 2) = 0$ and thus $F(1.01, 1.98) \approx 0 - 2 \times 0.01 + 16 \times 0.02 = 0.3$. 

\[ \text{In this case } F(1, 2) = 0 \text{ and thus } F(1.01, 1.98) \approx 0 - 2 \times 0.01 + 16 \times 0.02 = 0.3. \]
12. Let $f(x, y)$ be a differentiable function, and let $u = x + y$ and $v = x - y$. Find a constant $\alpha$ such that 

$$(f_x)^2 + (f_y)^2 = \alpha((f_u)^2 + (f_v)^2).$$

Solution: By the chain rule

$$(f_x)^2 + (f_y)^2 = (f_u + f_v)^2 + (f_u - f_v)^2 = 2((f_u)^2 + (f_v)^2).$$

Thus $\alpha = 2$.

13. Find the directional derivative $D_\vec{u}f$ at the given point in the direction indicated by the angle

(a) $f(x, y) = \sqrt{5x - 4y}$, $(2, 1)$, $\theta = -\pi/6$.

(b) $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \pi/3$.

Solution:

(a) $D_\vec{u}f = \left( \frac{5}{2\sqrt{6}} - \frac{2}{\sqrt{6}} \right) \cdot \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \frac{5}{4\sqrt{2}} + \frac{1}{\sqrt{6}}$.

(b) $D_\vec{u}f = 4\vec{j} \cdot \left( \frac{1}{2} + \frac{\sqrt{3}}{2} \vec{j} \right) = 2\sqrt{3}$.

14. Compute the directional derivatives $D_\vec{u}f$, where:

(a) $f(x, y) = \ln(x^2 + y^2)$, $\vec{u}$ is the unit vector pointing from $(0, 0)$ to $(1, 2)$.

(b) $f(x, y, z) = \sqrt{x^2 + 2y^2 + 3z^2}$, $\vec{u} = <1/\sqrt{2}, 1/\sqrt{2}, 0>$.

Solution:

(a) $D_\vec{u}f = \frac{2x}{x^2 + y^2} \frac{1}{\sqrt{5}} + \frac{2y}{x^2 + y^2} \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}} \frac{2x + 4y}{x^2 + y^2}$.

(b) $D_\vec{u}f = \frac{x}{\sqrt{2}(x^2 + 2y^2 + 3z^2)^{3/2}} - \frac{2y}{\sqrt{2}(x^2 + 2y^2 + 3z^2)^{3/2}}$

15. Find all points $(x, y, z)$ such that $D_\vec{u}f(x, y, z) = 0$, where $\vec{u} = <a, b, c>$ is a unit vector and $f(x, y, z) = \sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}$.

Solution:

$$D_\vec{u}f = \frac{a\alpha x + b\beta y + c\gamma z}{\sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}} = 0 \iff a\alpha x + b\beta y + c\gamma z = 0.$$

16. Compute the cosine of the angle between the gradient $\nabla f$ and the positive direction of the z-axis, where $f(x, y, z) = x^2 + y^2 + z^2$.

Solution: For $\cos \theta = \frac{(\nabla f) \cdot \vec{k}}{|\nabla f|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$.
17. The temperature at a point \((x, y, z)\) is given by \(T(x, y, z) = 200e^{-x^2-3y^2-9z^2}\).

(a) Find the rate of change of temperature at the point \(P(2, -1, 2)\) in the direction towards the point \((3, -3, 3)\).

(b) In which direction does the temperature increase the fastest at \(P\)?

(c) Find the maximum rate of increase at \(P\).

Solution:

(a) 
\[
D_uT = \nabla T \cdot \frac{1}{\sqrt{6}} < 1, -2, 1 > = -\frac{400}{e^{x^2+3y^2+9z^2}} < x, 3y, 9z > \cdot \frac{1}{\sqrt{6}} < 1, -2, 1 >
\]
\[
= -\frac{400}{e^{43}\sqrt{6}}(x - 6y + 9z) = -\frac{400 \times 26}{e^{43}\sqrt{6}} = -\frac{10400}{e^{43}\sqrt{6}}.
\]

(b) In the direction of the gradient. A unit vector pointing in the direction of \(\nabla T\) at the point \((2, -1, 2)\) is \(\vec{u} = -\frac{1}{\sqrt{337}} < 2, -3, 18 >\).

(c) The maximum rate of increase of \(T(x, y, z)\) at the point \((2, -1, 2)\) is 
\[
|\nabla T| = \frac{400 \times \sqrt{337}}{e^{43}}.
\]