HOMEWORK ASSIGNMENT #2, Math 253

1. Find the equation of a sphere if one of its diameters has end points $(1,0,5)$ and $(5,-4,7)$.

2. Find vector, parametric, and symmetric equations of the following lines.
   
   (a) the line passing through the points $(3,1,\frac{1}{2})$ and $(4,-3,3)$
   (b) the line passing through the origin and perpendicular to the plane $2x - 4y = 9$
   (c) the line lying on the planes $x + y - z = 2$ and $3x - 4y + 5z = 6$

3. Find the equation of the following planes.
   
   (a) the plane passing through the points $(-1,1,-1), (1,-1,2),$ and $(4,0,3)$
   (b) the plane passing through the point $(0,1,2)$ and containing the line $x = y = z$
   (c) the plane containing the lines $L_1: x = 1 + t, \quad y = 2 - t, \quad z = 4t$
   $\quad L_2: x = 2 - s, \quad y = 1 + 2s, \quad z = 4 + s$

4. Find the intersection of the line $x = t, \quad y = 2t, \quad z = 3t$, and the plane $x + y + z = 1$.

5. Find the distance between the point $(2,8,5)$ and the plane $x - 2y - 2z = 1$.

6. Show that the lines
   
   $L_1: \frac{x - 4}{2} = \frac{y + 5}{4} = \frac{z - 1}{-3}$
   $L_2: \frac{x - 2}{1} = \frac{y + 1}{3} = \frac{z}{2}$

   are skew. Find the distance between the two lines.

7. Identify and sketch the following surfaces.

   (a) $4x^2 + 9y^2 + 36z^2 = 36$
   (b) $4z^2 - x^2 - y^2 = 1$
   (c) $y^2 = x^2 + z^2$
   (d) $x^2 + 4z^2 - y = 0$
   (e) $y^2 + 9z^2 = 9$
   (f) $y = z^2 - x^2$

8. Find the polar equation for the curve represented by the following Cartesian equation.

   (a) $x = 4$
(b) $x^2 + y^2 = -2x$
(c) $x^2 - y^2 = 1$

9. Sketch the curve of the following polar equations.

   (a) $r = 5$
   (b) $\theta = \frac{3\pi}{4}$
   (c) $r = 2 \sin \theta$
   (d) $r = 3(1 - \cos \theta)$

10. (a) Change $(3, \frac{\pi}{3}, 1)$ from cylindrical to rectangular coordinates
    (b) Change $(\sqrt{3}, 1, 4)$ from rectangular to cylindrical coordinates
    (c) Change $(\sqrt{3}, 1, 2\sqrt{3})$ from rectangular to spherical coordinates
    (d) Change $(4, \frac{\pi}{4}, \frac{\pi}{3})$ from spherical to cylindrical coordinates
SOLUTIONS TO HOMEWORK ASSIGNMENT #2, Math 253

1. Find the equation of a sphere if one of its diameters has end points $(1, 0, 5)$ and $(5, -4, 7)$.

Solution:
The length of the diameter is \[ \sqrt{(5 - 1)^2 + (-4 - 0)^2 + (7 - 5)^2} = \sqrt{36} = 6, \] so the radius is 3. The centre is at the midpoint \( \left( \frac{1+5}{2}, \frac{0-4}{2}, \frac{5+7}{2} \right) = (3, -2, 6) \). Hence, the sphere is given as \( (x - 3)^2 + (y + 2)^2 + (z - 6)^2 = 9 \).

2. Find vector, parametric, and symmetric equations of the following lines.

(a) the line passing through the points $(3, 1, \frac{1}{2})$ and $(4, -3, 3)$

Solution:
The vector between two points is \( \vec{v} = \langle 4 - 3, -3 - 1, 3 - \frac{1}{2} \rangle = \langle 1, -4, \frac{5}{2} \rangle \). Hence the equation of the line is

Vector form: \( \vec{r} = \vec{r}_0 + t\vec{v} = \langle 4, -3, 3 \rangle + t\langle 1, -4, \frac{5}{2} \rangle = \langle 4 + t, -3 - 4t, 3 + \frac{5}{2}t \rangle \)

Parametric form: \( x = 4 + t, \quad y = -3 - 4t, \quad z = 3 + \frac{5}{2}t \)

Symmetric form: Solving the parametric form for \( t \) gives \( x - 4 = \frac{y + 3}{4} = \frac{z - 2}{5/2} \)

(b) the line passing through the origin and perpendicular to the plane \( 2x - 4y = 9 \)

Solution:
Perpendicular to the plane \( \Rightarrow \) parallel to the normal vector \( \vec{n} = \langle 2, -4, 0 \rangle \). Hence

Vector form: \( \vec{r} = \langle 0, 0, 0 \rangle + t\langle 2, -4, 0 \rangle = \langle 2t, -4t, 0 \rangle \)

Parametric form: \( x = 2t, \quad y = -4t, \quad z = 0 \)

Symmetric form \( \frac{x}{2} = \frac{y}{4}, \quad z = 0 \)

(c) the line lying on the planes \( x + y - z = 2 \) and \( 3x - 4y + 5z = 6 \)

Solution:
We can find the intersection (the line) of the two planes by solving \( z \) in terms of \( x \), and in terms of \( y \).

\[
\begin{align*}
(1) \quad & x + y - z = 2 \\
(2) \quad & 3x - 4y + 5z = 6
\end{align*}
\]

Solve \( z \) in terms of \( y \): \( 3 \times (1) - (2) \Rightarrow 7y - 8z = 0 \Rightarrow z = \frac{7}{8}y \)

Solve \( z \) in terms of \( x \): \( 4 \times (1) + (2) \Rightarrow 7x + z = 14 \Rightarrow z = 14 - 7x \)

Hence, symmetric form: \( 14 - 7x = \frac{7}{8}y = z \)

Set the symmetric form = \( t \), we have parametric form: \( x = \frac{14 - t}{7}, \quad y = \frac{8}{7}t, \quad z = t \)

Vector form: \( \vec{r} = \langle \frac{14 - t}{7}, \frac{8}{7}t, t \rangle \)
3. Find the equation of the following planes.

(a) the plane passing through the points \((-1, 1, -1), (1, -1, 2), \) and \((4, 0, 3)\)

**Solution:**

Name the points \(P(-1, 1, -1), Q(1, -1, 2), \) and \(R(4, 0, 3)\). Set up two vectors:

\[
\vec{v}_1 = \overrightarrow{PQ} = \langle 2, -2, 3 \rangle \quad (1)
\]

\[
\vec{v}_2 = \overrightarrow{PR} = \langle 5, -1, 4 \rangle \quad (2)
\]

Choose the normal vector \(\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -5, 7, 8 \rangle\). Hence the equation of the plane is \(-5(x + 1) + 7(y - 1) + 4(z + 1) = 0\) using point \(P\).

(b) the plane passing through the point \((0, 1, 2)\) and containing the line \(x = y = z\)

**Solution:**

Name \(Q(0, 1, 2)\). The line can be represented as \(\vec{r} = \langle t, t, t \rangle\), which crosses the point \(P(0, 0, 0)\) and is parallel to \(\vec{v} = \langle 1, 1, 1 \rangle\). Set \(\vec{b} = \overrightarrow{PQ} = \langle 0, 1, 2 \rangle\). Choose \(\vec{n} = \vec{v} \times \vec{b} = \langle 1, -2, 1 \rangle\) and hence the equation of the plane is \(x - 2y + z = 0\) using point \(P\).

(c) the plane containing the lines

\[L_1 : x = 1 + t, \quad y = 2 - t, \quad z = 4t\]

\[L_2 : x = 2 - s, \quad y = 1 + 2s, \quad z = 4 + s\]

**Solution:**

From \(L_1\) and \(L_2\), \(\vec{v}_1 = \langle 1, -1, 4 \rangle\) and \(\vec{v}_2 = \langle -1, 2, 1 \rangle\). Choose \(\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -9, -5, 1 \rangle\). Since \(L_1\) crosses the point \((1, 2, 0)\), the equation of the plane is \(-9(x - 1) - 5(y - 2) + z = 0\).

4. Find the intersection of the line \(x = t, \ y = 2t, \ z = 3t\), and the plane \(x + y + z = 1\).

**Solution:**

Substitute the line into the plane: \(t + 2t + 3t = 1 \Rightarrow t = \frac{1}{6}\).

Put \(t\) back to the line: \(x = \frac{1}{6}, \ y = \frac{1}{3}, \ z = \frac{1}{2}\).

Hence the intersection point is \(\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)\).

5. Find the distance between the point \((2, 8, 5)\) and the plane \(x - 2y - 2z = 1\).

**Solution:**

Name \(Q(2, 8, 5)\). Choose any point on the plane, say a convenient one \((x, 0, 0)\). So \(x - 2(0) - 2(0) = 1 \Rightarrow x = 1 \Rightarrow P(1, 0, 0)\). Then \(\vec{b} = \overrightarrow{PQ} = \langle 1, 8, 5 \rangle\). The normal vector of the plane is \(\vec{n} = \langle 1, -2, -2 \rangle\). The distance between the plane and the point is given as
distance \[= \left| \text{proj}_{\vec{n}} \vec{b} \right| = \left| \frac{\vec{n} \cdot \vec{b}}{\left| \vec{n} \right|} \right| = \frac{| -25 |}{3} = \frac{25}{3} \]

6. Show that the lines

\[L_1 : \frac{x - 4}{2} = \frac{y + 5}{4} = \frac{z - 1}{-3}\]
\[L_2 : \frac{x - 2}{1} = \frac{y + 1}{3} = \frac{z}{2}\]

are skew.

**Solution:**

Write the equation in parametric form.

\[L_1 : x = 2t + 4, \quad y = 4t - 5, \quad z = -3t + 1\]
\[L_2 : x = s + 2, \quad y = 3s - 1, \quad z = 2s\]

The lines are not parallel since the vectors \(\vec{v}_1 = \langle 2, 4, -3 \rangle\) and \(\vec{v}_2 = \langle 1, 3, 2 \rangle\) are not parallel. Next we try to find intersection point by equating \(x, y,\) and \(z\).

\[\begin{align*}
(1) \quad 2t + 4 &= s + 2 \\
(2) \quad 4t - 5 &= 3s - 1 \\
(3) \quad -3t + 1 &= 2s
\end{align*}\]

(1) gives \(s = 2t + 2\). Substituting into (2) gives \(4t - 5 = 3(2t + 2) - 1 \Rightarrow t = -5\). Then \(s = -8\). However, this contradicts with (3). So there is no solution for \(s\) and \(t\). Since the two lines are neither parallel nor intersecting, they are skew lines.

7. Identify and sketch the following surfaces.

(a) \(4x^2 + 9y^2 + 36z^2 = 36\)

**Solution:**

xy-plane: \(4x^2 + 9y^2 = 36\) ellipse
xz-plane: \(4x^2 + 36z^2 = 36\) ellipse
yz-plane: \(9y^2 + 36z^2 = 36\) ellipse
⇒ **ellipsoid**

(b) \(4z^2 - x^2 - y^2 = 1\)

**Solution:**

xy-plane: \(-x^2 - y^2 = 1\) nothing, try \(z = \text{constants}\)
\(z = c: -x^2 - y^2 = 1 - 4c^2 \Rightarrow x^2 + y^2 = 4c^2 - 1\) circles when \(4c^2 - 1 > 0\)
xz-plane: \(4z^2 - x^2 = 1\) hyperbola opening in \(z\)-direction
yz-plane: \(4z^2 - y^2 = 1\) hyperbola opening in \(z\)-direction
⇒ **hyperboloid of two sheets**
(c) \( y^2 = x^2 + z^2 \)

Solution:
- \( xy \)-plane: \( y^2 = x^2 \) cross
- \( xz \)-plane: \( 0 = x^2 + z^2 \) point at origin, try \( y = \) constants
- \( y = c: \ c^2 = x^2 + z^2 \) circles
- \( yz \)-plane: \( y^2 = z^2 \) cross
⇒ **cone**

(d) \( x^2 + 4z^2 - y = 0 \)

Solution:
- \( xy \)-plane: \( x^2 - y = 0 \) ⇒ \( y = x^2 \) parabola opening in \(+y\)-direction
- \( xz \)-plane: \( x^2 + 4z^2 = 0 \) point at origin, try \( y = \) constants
- \( y = c: \ c^2 + 4z^2 - c = 0 \) ⇒ \( x^2 + 4z^2 = c \) ellipses when \( c > 0 \)
- \( yz \)-plane: \( 4z^2 - y = 0 \) ⇒ \( y = 4z^2 \) parabola opening in \(+y\)-direction
⇒ **elliptic paraboloid**

(e) \( y^2 + 9z^2 = 9 \)

Solution:
- \( x \) missing: cylinder along \( x \)-direction
- \( yz \)-plane: \( y^2 + 9z^2 = 9 \) ellipse
⇒ **elliptic cylinder**

(f) \( y = z^2 - x^2 \)

Solution:
- \( xy \)-plane: \( y = z^2 \) parabola opening in \(+y\)-direction
- \( xz \)-plane: \( 0 = z^2 - x^2 \) ⇒ \( z^2 = x^2 \) cross, try \( y = \) constants
- \( y = c: \ c = z^2 - x^2 \) hyperbola opening in \( z \)-direction when \( c > 0 \), in \( x \)-direction when \( c < 0 \)
- \( yz \)-plane: \( y = -x^2 \) parabola opening in \(-y\)-direction
⇒ **hyperbolic paraboloid**

8. Find the polar equation for the curve represented by the following Cartesian equation.

(a) \( x = 4 \)

Solution:
\[ x = 4 \Rightarrow r \cos \theta = 4 \Rightarrow r = 4 \sec \theta \]

(b) \( x^2 + y^2 = -2x \)

Solution:
\[ x^2 + y^2 = -2x \Rightarrow r^2 = -2r \cos \theta \Rightarrow r = -2 \cos \theta \]
(c) \( x^2 - y^2 = 1 \)

\textbf{Solution:}
\[
x^2 - y^2 = 1 \Rightarrow r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1 \Rightarrow r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \Rightarrow r^2 \cos 2\theta = 1
\]
\[
\Rightarrow r^2 = \sec 2\theta \Rightarrow r = \pm \sqrt{\sec 2\theta}
\]

9. Sketch the curve of the following polar equations.

(a) \( r = 5 \)
(b) \( \theta = \frac{3\pi}{4} \)
(c) \( r = 2 \sin \theta \)
(d) \( r = 3(1 - \cos \theta) \)

10. (a) Change \( (3, \frac{\pi}{3}, 1) \) from cylindrical to rectangular coordinates

\textbf{Solution:}
\[
x = r \cos \theta = 3 \cos \frac{\pi}{3} = \frac{3}{2}, \ y = r \sin \theta = 3 \sin \frac{\pi}{3} = \frac{3\sqrt{3}}{2}, \ z = 1.
\]
Hence \( (x, y, z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 1\right) \)

(b) Change \( (\sqrt{3}, 1, 4) \) from rectangular to cylindrical coordinates

\textbf{Solution:}
\[
r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2, \ \tan \theta = \frac{\frac{3}{2}}{\frac{3}{2}} = 1 \Rightarrow \theta = \frac{\pi}{4} \text{ in first quadrant}, \ z = 4.
\]
Hence \( (r, \theta, z) = (2, \frac{\pi}{4}, 4) \)
(c) Change $(\sqrt{3}, 1, 2\sqrt{3})$ from rectangular to spherical coordinates

Solution:
\[
\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3 + 1 + 12} = 4, \quad \tan \theta = \frac{\rho}{x} = \frac{4}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \text{ in first quadrant}, \quad \phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1} \frac{2\sqrt{3}}{4} = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}.
\]
Hence $(\rho, \theta, \phi) = \left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$

(d) Change $(4, \frac{\pi}{4}, \frac{\pi}{3})$ from spherical to cylindrical coordinates

Solution:
\[
r = \rho \sin \phi = 4 \sin \frac{\pi}{3} = 2\sqrt{3}, \quad \theta = \frac{\pi}{4}, \quad z = \rho \cos \phi = 4 \cos \frac{\pi}{3} = 2.
\]
Hence $(r, \theta, z) = \left(2\sqrt{3}, \frac{\pi}{4}, 2\right)$