Math 200 Problem Set III

1) A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying
\[
\frac{d^2 \vec{r}}{dt^2} = -g \hat{k} - \alpha \frac{d\vec{r}}{dt}
\]
where \(\alpha\) is a positive constant. If \(\vec{r} = \vec{r}_0\) and \(\frac{d\vec{r}}{dt} = \vec{v}_0\) at time \(t = 0\), find \(\vec{r}(t)\). (Hint: Define \(\vec{u}(t) = e^{\alpha t} \frac{d\vec{r}}{dt}(t)\) and substitute \(\frac{d\vec{r}}{dt}(t) = e^{-\alpha t} \vec{u}(t)\) into the given differential equation to find a differential equation for \(\vec{u}\).)

2) A gun fires a shell with a muzzle speed of 150 m/s. While the shell is in the air, it experiences a downward (vertical) gravitational acceleration of 9.8 m/s\(^2\) and an eastward (horizontal) Coriolis acceleration of 5 cm/s\(^2\). Air resistance may be ignored. The target is 1500 m due north of the gun and both the gun and target are at sea level. For which initial velocities will the shell hit the target?

3) Find the specified parametrization of the first quadrant part of the circle \(x^2 + y^2 = a^2\).
   a) In terms of the \(y\) coordinate.
   b) In terms of the angle between the tangent line and the positive \(x\)-axis.
   c) In terms of the arc length from \((0, a)\).

4) Evaluate, if possible,
   a) \(\lim_{(x,y) \to (2,-1)} xy + x^2\)
   b) \(\lim_{(x,y) \to (0,0)} \frac{x}{x^2 + y^2}\)
   c) \(\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}\)
   d) \(\lim_{(x,y) \to (0,0)} \frac{x^3}{x^2 + y^2}\)

5) Find all first partial derivatives of the following functions and evaluate them at the given point.
   a) \(f(x, y, z) = x^3 y^4 z^5\) \((0, -1, -1)\)
   b) \(w(x, y, z) = \ln(1 + e^{xyz})\) \((2, 0, -1)\)
   c) \(f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}\) \((-3, 4)\)

6) Use the definition of the derivative to evaluate \(f_x(0, 0)\) and \(f_y(0, 0)\) for
\[
f(x, y) = \begin{cases} 
  \frac{x^2 - 2y^2}{x - y} & \text{if } x \neq y \\
  0 & \text{if } x = y
\end{cases}
\]

7) Find an approximate value for \(f(x, y) = \sin(\pi xy + \ln y)\) at \((0.01, 1.05)\) without using a calculator or computer.

8) Let \((r, \theta)\) and \((x, y)\) be polar and Cartesian coordinates in the plane. Use a geometrical argument and the definition of partial derivative to decide whether or not \(\frac{\partial r}{\partial x}\) and \(\left(\frac{\partial x}{\partial r}\right)^{-1}\) are equal.
MATH 200 PROBLEM SET III SOLUTIONS

1) A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2 \vec{r}}{dt^2} = -g\hat{k} - \alpha \frac{d\vec{v}}{dt}$$

where $\alpha$ is a positive constant. If $\vec{r} = \vec{r}_0$ and $\frac{d\vec{r}}{dt} = \vec{v}_0$ at time $t = 0$, find $\vec{r}(t)$. (Hint: consider $\vec{u}(t) = e^{\alpha t} \frac{d\vec{r}}{dt}(t).$)

Solution. Define $\vec{u}(t) = e^{\alpha t} \frac{d\vec{r}}{dt}(t)$. Then

$$\frac{d\vec{u}}{dt}(t) = \alpha e^{\alpha t} \frac{d\vec{r}}{dt}(t) + e^{\alpha t} \frac{d^2\vec{r}}{dt^2}(t) = \alpha e^{\alpha t} \frac{d\vec{r}}{dt}(t) - g e^{\alpha t} \hat{k} - \alpha e^{\alpha t} \frac{d\vec{v}}{dt}(t) = -g e^{\alpha t} \hat{k}$$

Integrating both sides of this equation from $t = 0$ to $t = T$ gives

$$\vec{u}(T) - \vec{u}(0) = -g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} \Rightarrow \vec{u}(T) = \vec{u}(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} = \frac{d\vec{r}}{dt}(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k}$$

Subbing in $\vec{u}(T) = e^{\alpha t} \frac{d\vec{r}}{dt}(T)$ and multiplying through by $e^{-\alpha T}$

$$\frac{d\vec{r}}{dt}(T) = e^{-\alpha T} \vec{v}_0 - g \frac{1 - e^{-\alpha T}}{\alpha} \hat{k}$$

Integrating both sides of this equation from $T = 0$ to $T = t$ gives

$$\vec{r}(t) - \vec{r}(0) = \frac{e^{-\alpha t} - 1}{\alpha} \vec{v}_0 - g \frac{1 - e^{-\alpha t}}{\alpha} \hat{k} + g \frac{e^{-\alpha t} - 1}{\alpha} \hat{k}$$

$$\Rightarrow \vec{r}(t) = \vec{r}_0 - \frac{e^{-\alpha t} - 1}{\alpha} \vec{v}_0 + g \frac{1 - e^{-\alpha t} - e^{-\alpha t}}{\alpha^2} \hat{k}$$

2) A gun fires a shell with a muzzle speed of 150 m/s. While the shell is in the air, it experiences a downward (vertical) gravitational acceleration of 9.8 m/s$^2$ and an eastward (horizontal) Coriolis acceleration of 5cm/s$^2$. Air resistance may be ignored. The target is 1500 m due north of the gun and both the gun and target are at sea level. For which initial velocities will the shell hit the target?

Solution. Choose a coordinate system in which, the gun is at the origin, the $x$–axis points due east, the $y$–axis points due north and the $z$–axis points straight up. If the shell is fired a time zero with initial velocity $\vec{v} = (v_x, v_y, v_z)$, then its position $\vec{r}(t)$ at time $t$ obeys

$$\vec{r}''(t) = -9.8 \hat{k} + 0.05 \hat{i}$$

$$\vec{r}'(t) = \vec{v} - 9.8t \hat{k} + 0.05t \hat{i}$$

$$\vec{r}(t) = \vec{v}t - 4.9t^2 \hat{k} + 0.025t^2 \hat{i}$$

The shell lands at the nonzero time for which the $z$–component of $t\vec{v} - 4.9t^2 \hat{k} + 0.025t^2 \hat{i}$, namely, $tv_z - 4.9t^2$ is zero. So the shell lands at $t = \frac{v_z}{4.9}$. At this time, the position of the shell, $\frac{tv_x}{4.9} \hat{i} + 0.025(\frac{tv_z}{4.9})^2 \hat{i} + 7350 \frac{v_y}{v_z}$, is to be 1500$\hat{j}$. Hence

$$\frac{tv_x}{4.9} + 0.025(\frac{tv_z}{4.9})^2 = 0 \Rightarrow v_x \frac{tv_x}{4.9} + 1500 = \Rightarrow v_x = -0.025 v_z \Rightarrow v_x = 1500 \times 4.9 \frac{1}{v_z} \Rightarrow 7350 \frac{v_y}{v_z}$$

In order for the muzzle speed to be 150 m/s, we also need

$$v_x^2 + v_y^2 + v_z^2 = 150^2 \Rightarrow (1 + \frac{1}{190})v_x^2 + 7350^2 \frac{v_z}{v_x} = 150^2 \Rightarrow (1 + \frac{1}{190})v_y^2 - 150^2 v_x^2 + 7350^2 = 0$$

$$\Rightarrow v_x^2 = \frac{150^2 \pm \sqrt{150^4 - 4 \times 7350^2(1 + \frac{1}{190})}}{2(1 + \frac{1}{190})} = 2732.97 \text{ or } 19766.44$$
Using \( v_x = \frac{1}{\tan \alpha} v_z \), \( v_y = 7350 \frac{1}{v_z} \), gives

\[
(v_x, v_y, v_z) = (-0.27, 140.60, 52.28) \text{ or } (-0.72, 52.28, 140.59) \text{ m}
\]

3) Find the specified parametrization of the first quadrant part of the circle \( x^2 + y^2 = a^2 \).
   a) In terms of the \( y \) coordinate.
   b) In terms of the angle between the tangent line and the positive \( x \)-axis.
   c) In terms of the arc length from \((0, a)\).

**Solution.**

a) \( (x(t), y(t)) = (\sqrt{a^2 - t^2}, t), \ 0 \leq t \leq a \)

b) Let \( \theta \) be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive \( x \)-axis. The tangent line to the circle at \((a \cos \theta, a \sin \theta)\) is perpendicular to the radius vector and so makes angle \( \phi = \frac{\pi}{2} + \theta \) with the positive \( x \) axis. The desired parametrization is

\[
(x(\phi), y(\phi)) = (a \cos(\phi - \frac{\pi}{2}), a \sin(\phi - \frac{\pi}{2})) = (a \sin \phi, -a \cos \phi), \ \frac{\pi}{2} \leq \phi \leq \pi
\]

c) Let \( \theta \) be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive \( x \)-axis. The arc from \((0, a)\) to \((a \cos \theta, a \sin \theta)\) subtends an angle \( \frac{\pi}{2} - \theta \) and so has length \( s = a(\frac{\pi}{2} - \theta) \). Thus \( \theta = \frac{\pi}{2} - \frac{s}{a} \) and the desired parametrization is

\[
(x(\phi), y(\phi)) = (a \cos(\frac{\pi}{2} - \frac{s}{a}), a \sin(\frac{\pi}{2} - \frac{s}{a})), \ 0 \leq s \leq \frac{\pi}{2}a
\]

4) Evaluate, if possible,

a) \( \lim_{(x,y)\to(2,-1)} xy + x^2 = 2(-1) + 2^2 = 2 \)

b) \( \lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2} \) undefined

c) \( \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = \lim_{r \to 0} \frac{r \cos \theta}{r^2} = \lim_{r \to 0} \frac{\cos \theta}{r} = \text{undefined} \)

d) \( \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \to 0} r \cos^3 \theta = 0 \)

5) Find all first partial derivatives of the following functions and evaluate them at the given point.

a) \( f(x, y, z) = x^3 y^4 z^5 \) \( (0, -1, -1) \)

b) \( w(x, y, z) = \ln (1 + e^{xyz}) \) \( (2, 0, -1) \)

c) \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \) \( (-3, 4) \)

**Solution.**

a) \[
\begin{align*}
  f_x(x, y, z) &= 3x^2 y^4 z^5 \quad f_x(0, -1, -1) = 0 \\
  f_y(x, y, z) &= 4x^3 y^3 z^5 \quad f_y(0, -1, -1) = 0 \\
  f_z(x, y, z) &= 5x^3 y^4 z^4 \quad f_z(0, -1, -1) = 0
\end{align*}
\]
\[ w_x(x, y, z) = \frac{2xe^{xyz}}{1 + e^{xyz}} \quad w_x(2, 0, -1) = 0 \]
\[ w_y(x, y, z) = \frac{2xe^{xyz}}{1 + e^{xyz}} \quad w_y(2, 0, -1) = 1 \]
\[ w_z(x, y, z) = \frac{2ye^{xyz}}{1 + e^{xyz}} \quad w_z(2, 0, -1) = 0 \]

b) Let \( w(x, y, z) \) be polar and Cartesian coordinates in the plane. Use a geometrical argument and the definition of partial derivative to decide whether or not \( \frac{\partial w}{\partial x} \) and \( \left( \frac{\partial w}{\partial x} \right)^{-1} \) are equal.

\[ f_x(x, y) = -\frac{x}{(x^2 + y^2)^{3/2}} \quad f_x(-3, 4) = \frac{3}{125} \]
\[ f_y(x, y) = -\frac{y}{(x^2 + y^2)^{3/2}} \quad f_y(-3, 4) = -\frac{4}{125} \]

6) Use the definition of the derivative to evaluate \( f_x(0, 0) \) and \( f_y(0, 0) \) for
\[ f(x, y) = \begin{cases} \frac{x^2 - 2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \]

**Solution.** By definition
\[ f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0)-f(x_0, y_0)}{\Delta x} \quad f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y)-f(x_0, y_0)}{\Delta y} \]

Setting \( x_0 = y_0 = 0 \),
\[ f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0)-f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x^2 - 2 \cdot 0^2)/(\Delta x - 0)}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1 \]
\[ f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y)-f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(0^2 - 2 \cdot 0^2)/(0 - \Delta y)}{\Delta y} = \lim_{\Delta y \to 0} 2 = 2 \]

7) Find an approximate value for \( f(x, y) = \sin(\pi xy + \ln y) \) at \((0.01, 1.05)\) without using a calculator or computer.

**Solution.** Apply \( f(0.01, 1.05) \approx f(0, 1) + f_x(0, 1)(0.01) + f_y(0, 1)(0.05) \), with
\[ f(x, y) = \sin(\pi xy + \ln y) \quad f(0, 1) = \sin 0 = 0 \]
\[ f_x(x, y) = \pi y \cos(\pi xy + \ln y) \quad f_x(0, 1) = \pi \cos 0 = \pi \]
\[ f_y(x, y) = (\pi x + \frac{1}{y}) \cos(\pi xy + \ln y) \quad f_x(0, 1) = \cos 0 = 1 \]

This gives
\[ f(0.01, 1.05) \approx f(0, 1) + f_x(0, 1)(0.01) + f_y(0, 1)(0.05) = 0 + \pi(0.01) + 1(0.05) \approx 0.0814 \]

8) Let \((r, \theta)\) and \((x, y)\) be polar and Cartesian coordinates in the plane. Use a geometrical argument and the definition of partial derivative to decide whether or not \( \frac{\partial w}{\partial r} \) and \( \left( \frac{\partial w}{\partial r} \right)^{-1} \) are equal.

**Solution.** In computing \( \frac{\partial w}{\partial r} \), \( \theta \) is held fixed, \( r \) is changed by a small amount \( dr \) and the resulting \( dx \) is found as in the figure on the left below. In computing \( \frac{\partial w}{\partial \theta} \), \( y \) is held fixed, \( x \) is changed by a small amount \( dx \) and the resulting \( dr \) is found as in the figure on the right below.
Here are the two figures combined together. I have arranged that the same $dr$ is used in both computations. In order for the $dr$’s to be the same in both computations, the two $dx$’s have to be different (unless $\theta = 0$). So $\frac{\partial x}{\partial r} \neq \frac{\partial r}{\partial x}$. 