Math 200 Problem Set II

1) Find the equation of the sphere which has the two planes \( x + y + z = 3, \ x + y + z = 9 \) as tangent planes if the centre of the sphere is on the planes \( 2x - y = 0, \ 3x - z = 0 \).

2) Find the equation of the plane that passes through the point \((-2, 0, -1)\) and through the line of intersection of \( 2x + 3y - z = 0, \ x - 4y + 2z = -5 \).

3) Find the equations of the line through \((2, -1, -1)\) and parallel to each of the two planes \( x + y = 0 \) and \( x - y + 2z = 0 \). Express the equations of the line in vector and scalar parametric forms and in symmetric form.

4) Sketch and describe the following surfaces.
   a) \( 4x^2 + y^2 = 16 \)  
   b) \( x + y + 2z = 4 \)  
   c) \( z^2 = y^2 + 4 \)  
   d) \( \frac{x}{4} = \frac{y^2}{2} + \frac{z^2}{9} \)  
   e) \( \frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16} \)  
   f) \( z = x^2 \)  
   g) \( z = \frac{y^4}{9} - \frac{x^2}{9} \)  
   h) \( y^2 = x^2 + z^2 \)  
   i) \( \frac{x^2}{9} + \frac{y^2}{12} + \frac{z^2}{9} = 1 \)  
   j) \( x^2 + y^2 + z^2 + 4x - by + 9z - b = 0 \) where \( b \) is a constant.

5) Sketch the graphs of
   a) \( f(x, y) = \sin x \) \( 0 \leq x \leq 2\pi, \ 0 \leq y \leq 1 \)  
   b) \( f(x, y) = \sqrt{x^2 + y^2} \)  
   c) \( f(x, y) = |x| + |y| \)

6) Sketch some of the level curves of
   a) \( f(x, y) = x^2 + 2y^2 \)  
   b) \( f(x, y) = xy \)  
   c) \( f(x, y) = \frac{y}{x^2+y^2} \)  
   d) \( f(x, y) = xe^{-y} \)

7) Describe the level surfaces of
   a) \( f(x, y, z) = x^2 + y^2 + z^2 \)  
   b) \( f(x, y, z) = x + 2y + 3z \)  
   c) \( f(x, y, z) = x^2 + y^2 \)

8) Find the velocity, speed and acceleration at time \( t \) of the particle whose position is \( \vec{r}(t) \). Describe the path of the particle.
   a) \( \vec{r}(t) = a \cos t \hat{i} + a \sin t \hat{j} + ct \hat{k} \)  
   b) \( \vec{r}(t) = a \cos t \sin t \hat{i} + a \sin^2 t \hat{j} + a \cos t \hat{k} \)
1) Find the equation of the sphere which has the two planes \( x + y + z = 3 \), \( x + y + z = 9 \) as tangent planes if the centre of the sphere is on the planes \( 2x - y = 0 \), \( 3x - z = 0 \).

**Solution.** The planes \( x + y + z = 3 \) and \( x + y + z = 9 \) are parallel. So the centre lies on \( x + y + z = 6 \) (the plane midway between \( x + y + z = 3 \) and \( x + y + z = 9 \)) as well as on \( y = 2x \) and \( z = 3x \). Solving,

\[
y = 2x, \quad z = 3x, \quad x + y + z = 6 \quad \Rightarrow \quad x + 2x + 3x = 6 \quad \Rightarrow \quad x = 1, \quad y = 2, \quad z = 3
\]

So the centre is at \((1, 2, 3)\). The normal to \( x + y + z = 3 \) is \((1, 1, 1)\). The points \((1, 1, 1)\) on \( x + y + z = 3 \) and \((3, 3, 3)\) on \( x + y + z = 9 \) differ by a vector, \((2, 2, 2)\), which is a multiple of this normal. So the distance between the planes is \( \left\lVert (2, 2, 2) \right\rVert = 2\sqrt{3} \) and the radius of the sphere is \( \sqrt{3} \). The sphere is

\[
(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3
\]

2) Find the equation of the plane that passes through the point \((-2, 0, -1)\) and through the line of intersection of \(2x + 3y - z = 0\), \(x - 4y + 2z = -5\).

**Solution.** First we’ll find two points on the line of intersection of \(2x + 3y - z = 0\), \(x - 4y + 2z = -5\). This will give us three points on the plane.

\[
\begin{align*}
2x + 3y - z &= 0 \\
x - 4y + 2z &= -5
\end{align*}
\]

\[
\Leftrightarrow \begin{align*}
2x + 3y &= z \\
x - 4y &= -2z - 5
\end{align*}
\]

In the last step, we subtracted twice the second equation from the first. So if \(z = -2\), then \(y = \frac{1}{2}\) and \(x = 0\). We conclude that the three points \((-2, 0, -1)\), \((-1, 0, -2)\) and \((0, -\frac{1}{2}, -\frac{1}{2})\) must all lie on the plane. So the two vectors \((-2, 0, -1) - (-1, 0, -2) = (-1, 0, 1)\) and \((0, -\frac{1}{2}, -\frac{1}{2}) - (-1, 0, -2) = (1, -\frac{1}{2}, -\frac{1}{2})\) must be parallel to the plane. So the normal to the plane is \(\langle -1, 0, 1 \rangle \times \langle 1, -\frac{1}{2}, -\frac{1}{2} \rangle = \langle \frac{1}{2}, -\frac{5}{2}, \frac{5}{2} \rangle\) or, equivalently \(\hat{n} = (5, -9, 5)\). The equation of the plane is

\[
5(x + 2) - 9y + 5(z + 1) = 0 \quad \text{or} \quad 5x - 9y + 5z = -15
\]

3) Find the equations of the line through \((2, -1, -1)\) and parallel to each of the two planes \(x + y = 0\) and \(x - y + 2z = 0\). Express the equations of the line in vector and scalar parametric forms and in symmetric form.

**Solution.** One vector normal to \(x + y = 0\) is \((1, 1, 0)\). One vector normal to \(x - y + 2z = 0\) is \((1, -1, 2)\). The vector \((1, -1, -1)\) is perpendicular to both of those normals and hence is parallel to both planes. So \((1, -1, -1)\) is also parallel to the line. The vector parametric equation of the line is

\[
\vec{r} = (2, -1, -1) + t(1, -1, -1)
\]

The scalar parametric equations of the line are

\[
x = 2 + t, \quad y = -1 - t, \quad z = -1 - t
\]

The symmetric equations are

\[
t = \frac{x - 2}{-y + 1} = \frac{-z - 1}{1}
\]
4) Sketch and describe the following surfaces.

a) \(4x^2 + y^2 = 16\)  
   b) \(x + y + 2z = 4\)  
   c) \(z^2 = y^2 + 4\)  
   d) \(x = \frac{y^2}{4} + \frac{z^2}{9}\)  
   e) \(\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}\)  
   f) \(z = x^2\)  
   g) \(z = \frac{x^2}{4} - \frac{y^2}{9}\)  
   h) \(y^2 = x^2 + z^2\)  
   i) \(\frac{y^2}{9} + \frac{z^2}{4} = 1\)  
   j) \(x^2 + y^2 + z^2 + 4x - by + 9z - b = 0\) where \(b\) is a constant.

Solution.

a) This is an elliptic cylinder parallel to the \(z\)-axis. Its cross-section (parallel to the \(xy\)-plane) is an ellipse centered on the origin with one semiaxis of length 2 along the \(x\)-axis and one semiaxis of length 4 along the \(y\)-axis.

b) This is a plane through \((4, 0, 0)\), \((0, 4, 0)\) and \((0, 0, 2)\).

c) This is a hyperbolic cylinder parallel to the \(x\)-axis. Its cross-section (parallel to the \(yz\)-plane) is a hyperbola centered on the origin with asymptotes \(z = \pm y\). No points have \(|z| < 2\).

d) This is an elliptic paraboloid. Its cross-sections parallel to the \(yz\)-plane are ellipses with semiaxes \(\sqrt{x}\) along the \(y\)-axis and \(\frac{3\sqrt{2}}{2}\sqrt{x}\) along the \(z\)-axis. There are no points on the surface with \(x < 0\). As you move out along the \(x\)-axis, the ellipses grow at a rate proportional to \(\sqrt{x}\).

e) This is a hyperboloid of one sheet with axis the \(x\)-axis. Its cross-sections parallel to the \(yz\)-plane are ellipses with semiaxes \(3\sqrt{1 + \frac{x^2}{16}}\) parallel to the \(y\)-axis and \(2\sqrt{1 + \frac{x^2}{16}}\) parallel to the \(z\)-axis. As you move out along the \(x\)-axis, the ellipses grow at a rate proportional to \(\sqrt{1 + \frac{x^2}{16}}\), which for large \(x\) is approximately \(\frac{|x|}{4}\).

f) This is a parabolic cylinder parallel to the \(y\)-axis. Its cross-section (parallel to the \(xz\)-plane) is a parabola with vertex at the origin.

g) This is a hyperbolic paraboloid. Each cross-section parallel to the \(xz\)-plane is a parabola opening downwards, with vertex at \((0, y, \frac{y^2}{4})\). Each cross-section parallel to the \(yz\)-plane is a parabola opening upwards, with vertex at \((x, 0, -\frac{x^2}{9})\). Each cross-section parallel to the \(xy\)-plane is a hyperbola with asymptotes \(y = \pm \frac{x}{3}x\).

h) This is a circular cone. Each cross-section parallel to the \(xz\)-plane is a circle of radius \(|y|\).

i) This is an ellipsoid centered on the origin with semiaxes 3, \(2\sqrt{3}\) and 3 along the \(x\), \(y\) and \(z\)-axes, respectively.

j) \(x^2 + y^2 + z^2 + 4x - by + 9z - b = 0\) if and only if \((x + 2)^2 + (y - \frac{b}{2})^2 + (z + \frac{9}{2})^2 = b + 4 + \frac{b^2}{4} + \frac{81}{4}\). This is a sphere of radius \(\frac{1}{2}\sqrt{b^2 + 4b + 97}\) centered on \(\frac{1}{2}(-4, b, -9)\).
5) Sketch the graphs of
   a) \( f(x, y) = \sin x \quad 0 \leq x \leq 2\pi, \ 0 \leq y \leq 1 \)
   b) \( f(x, y) = \sqrt{x^2 + y^2} \)
   c) \( f(x, y) = |x| + |y| \)

   **Solution.** a) The graph is \( z = \sin x \) with \((x, y)\) running over \(0 \leq x \leq 2\pi, \ 0 \leq y \leq 1\). For each fixed \(y_0\) between 0 and 1, the intersection of this graph with the vertical plane \(y = y_0\) is the same \(\sin\) graph \(z = \sin x\) with \(x\) running from 0 to \(2\pi\). So the whole graph is just a bunch of 2–d \(\sin\) graphs stacked side–by–side. This gives the graph on the left below.

   b) The graph is \(z = \sqrt{x^2 + y^2}\). For each fixed \(z_0 \geq 0\), the intersection of this graph with the horizontal plane \(z = z_0\) is the circle \(\sqrt{x^2 + y^2} = z_0\). This circle is centred on the \(z\)–axis and has radius \(z_0\). So the graph is the upper half of a cone. It is the middle sketch below.

   c) The graph is \(z = |x| + |y|\). For each fixed \(z_0 \geq 0\), the intersection of this graph with the horizontal plane \(z = z_0\) is the square \(|x| + |y| = z_0\). The side of the square with \(x, y \geq 0\) is the straight line \(x + y = z_0\). The side of the square with \(x \geq 0\) and \(y \leq 0\) is the straight line \(x - y = z_0\) and so on. The four corners of the square are \((\pm z_0, 0, z_0)\) and \((0, \pm z_0, z_0)\). So the graph is a stack of squares. It is an upside down four–sided pyramid. The part of the pyramid in the first octant (that is, \(x, y, z \geq 0\)) is the right hand sketch below.

6) Sketch some of the level curves of
   a) \( f(x, y) = x^2 + 2y^2 \)
   b) \( f(x, y) = xy \)
   c) \( f(x, y) = \frac{y}{x^2+y^2} \)
   d) \( f(x, y) = xe^{-y} \)

   **Solution.**
Observe that, for \( c \neq 0 \) and \((x, y) \neq (0, 0)\),
\[
\frac{y}{x^2+y^2} = c \iff x^2 + y^2 - \frac{1}{c} y = 0 \iff x^2 + (y - \frac{1}{2c})^2 = \frac{1}{4c^2}
\]
is a circle that passes through the origin and has centre at \( (0, \frac{1}{2c}) \).

7) Describe the level surfaces of

a) \( f(x, y, z) = x^2 + y^2 + z^2 \)
b) \( f(x, y, z) = x + 2y + 3z \)
c) \( f(x, y, z) = x^2 + y^2 \)

Solution. a) If \( c > 0 \), \( f(x, y, z) = c \) is the sphere of radius \( \sqrt{c} \) centered at the origin. If \( c = 0 \), \( f(x, y, z) = c \) is just the origin. If \( c < 0 \), no \((x, y, z)\) satisfies \( f(x, y, z) = c \).
b) \( f(x, y, z) = c \) is the plane normal to \((1, 2, 3)\) passing through \((c, 0, 0)\).
c) If \( c > 0 \), \( f(x, y, z) = c \) is the cylinder parallel to the \( z \)-axis whose cross-section is the circle of radius \( \sqrt{c} \) centered at the origin. If \( c = 0 \), \( f(x, y, z) = c \) is the \( z \)-axis. If \( c < 0 \), no \((x, y, z)\) satisfies \( f(x, y, z) = c \).

8) Find the velocity, speed and acceleration at time \( t \) of the particle whose position is \( \vec{r}(t) \). Describe the path of the particle.

a) \( \vec{r}(t) = a \cos t \, \hat{i} + a \sin t \, \hat{j} + ct \, \hat{k} \)
b) \( \vec{r}(t) = a \cos t \sin t \, \hat{i} + a \sin^2 t \, \hat{j} + a \cos t \, \hat{k} \)

Solution. a)
\[
\vec{r}(t) = a \cos t \, \hat{i} + a \sin t \, \hat{j} + ct \, \hat{k} \\
\vec{v}(t) = -a \sin t \, \hat{i} + a \cos t \, \hat{j} + c \, \hat{k} \\
\vec{a}(t) = -a \cos t \, \hat{i} - a \sin t \, \hat{j}
\]
The \((x, y) = a(\cos t, \sin t)\) coordinates go around a circle of radius \( a \) and centre \((0, 0)\) counterclockwise. One circle is completed for each increase of \( t \) by \( 2\pi \). At the same time, the \( z \) coordinate increases at a constant rate. Each time the \((x, y)\) coordinates complete one circle, the \( z \) coordinate increases by \( 2\pi c \).
The path is a helix with radius \( a \) and with each turn having height \( 2\pi c \).
b)
\[
\vec{r}(t) = a \cos t \sin t \, \hat{i} + a \sin^2 t \, \hat{j} + a \cos t \, \hat{k} \\
= \frac{a}{2} \sin 2t \, \hat{i} + a \frac{1 - \cos 2t}{2} \, \hat{j} + a \cos t \, \hat{k} \\
\vec{v}(t) = a \cos 2t \, \hat{i} + a \sin 2t \, \hat{j} - a \sin t \, \hat{k} \\
\vec{a}(t) = -2a \sin 2t \, \hat{i} + 2a \cos 2t \, \hat{j} - a \cos t \, \hat{k}
\]
The \((x, y)\) coordinates go around a circle of radius \( \frac{a}{2} \) and centre \( \left(0, \frac{a}{2}\right)\) counterclockwise. At the same time the \( z \) coordinate oscillates half as fast.