Problem 1 (8 points): Find the work done by the force field
\[ \mathbf{F}(x, y) = x \mathbf{i} + (y + 2) \mathbf{j} \]
in moving an object along an arch of the cycloid
\[ \mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j}, \quad 0 \leq t \leq 2\pi. \]

\[
\int_{t=0}^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{2\pi} \left< t - \sin t, (1 - \cos t) + 2 \right> \cdot \left< 1 - \cos t, \sin t \right> dt
\]

\[
= \int_{t=0}^{2\pi} \left[ (t - \sin t)(1 - \cos t) + (3 - \cos t)\sin t \right] dt
\]

\[
= \int_{0}^{2\pi} \left[ t - \sin t - t\cos t + \sin t \cos t + 3\sin t - \sin t \cos t \right] dt
\]

\[
= \int_{0}^{2\pi} (2\sin t + t - t\cos t) dt
\]

\[
= \left[ -2\cos t + \frac{t^2}{2} - t\sin t - \cos t \right]_{0}^{2\pi} = \frac{4\pi^2}{2} = 2\pi^2
\]
Problem 2 (8 points): Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = (y^3 + \tan(x^2)) \hat{i} + (e^{\sqrt{y}} - x^3) \hat{j}$$

and $C$ is the circle $x^2 + y^2 = 4$, oriented in the counterclockwise direction.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

$$P = y^3 + \tan(x^2) \quad Q = e^{\sqrt{y}} - x^3$$

Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{x^2+y^2 \leq 4} (Q_x - P_y) \, dA$$

$$= \iint_{x^2+y^2 \leq 4} (-3x^2 - 3y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^2 -3r^2 \, rdrd\theta$$

$$= 2\pi \int_0^2 -3r^3dr = 2\pi \left(-\frac{3}{4}\right) \left[ r^4 \right]_0^2 = \frac{-6\pi}{4} \cdot 16$$

$$= -24\pi$$
Problem 3. (2 points each). Let $\vec{F}$ be the vector field

$$\vec{F} = x \hat{i} + y \hat{j} + z \hat{k},$$

let

$$r = |\vec{F}|,$$

and let $n$ be an integer. Compute and simplify the following. Express your answer in terms of $n$, $\vec{F}$, and $r$. CREDIT WILL BE GIVEN FOR THE ANSWERS ONLY, YOU MUST EXPRESS YOUR ANSWER IN TERMS OF $n$, $\vec{F}$, AND $r$ TO RECEIVE CREDIT. THERE SHOULD BE NO $x$’s, $y$’s, or $z$’s IN YOUR ANSWER, AND ALL DERIVATIVES (including grad, curl, and div) SHOULD BE CARRIED OUT IN FULL. YOUR ANSWER SHOULD BE AN ALGEBRAIC EXPRESSION OF $n$, $\vec{F}$, AND $r$. Symbols like “+$”, “-”, “(”, “)”, etc. are of course allowed. Numbers (“1”, “2”, “3”, etc.) are allowed. You may use the following product rule

$$\text{div}(f \vec{F}) = \text{grad} f \cdot \vec{F} + f \text{div}(\vec{F}).$$

1. $\text{div}\vec{F}$
   
   $$\text{div} (x \hat{i} + y \hat{j} + z \hat{k}) = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$$

2. $\text{grad}(r^n)$

   $$\nabla (r^{n})(x^2+y^2+z^2)^{n/2} = \frac{n}{2} x (x^2+y^2+z^2)^{n/2-1}, \quad \frac{n}{2} y (x^2+y^2+z^2)^{n/2-1}, \quad \frac{n}{2} z (x^2+y^2+z^2)^{n/2-1}$$

   $$= n (x^2+y^2+z^2)^{n/2-1} \langle x, y, z \rangle = \boxed{n \frac{r^{n-2}}{r}}$$

3. $\text{div}(r^n \vec{F}) = \text{grad}(r^n) \cdot \vec{F} + r^n \text{div} \vec{F}$

   $$= n r^{n-2} \hat{r} \cdot \vec{F} + 3r^n = \boxed{(3+n) r^n}$$

4. $\nabla^2(r^n) = \text{div}(\text{grad}(r^n)) = \text{div}(nr^{n-2} \vec{F}) = n (3 + n - 2) r^{n-2} = \boxed{n(n+1) r^{n-2}}$
Problem 4 (8 points).
Suppose we know that the vector field
\[ \mathbf{F} = (Axy - y^2 + y) \mathbf{i} + (3x^2 - Bxy + x) \mathbf{j} \]
is conservative where \( A \) and \( B \) are constants.

1. Find the values of \( A \) and \( B \).
2. Find a scalar function \( f(x, y) \) such that \( \mathbf{F} = \nabla f \).
3. Evaluate the line integral
   \[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
   where \( C \) is the curve from \((1,1)\) to \((\frac{1}{2}, 2)\) given by \( xy = 1 \).
4. Evaluate the line integral
   \[ \int_C (6xy - y^2 + y)dx + (3x^2 - 3xy + x)dy \]
   where \( C \) is the curve from \((1,1)\) to \((\frac{1}{2}, 2)\) given by \( xy = 1 \).

1. \( Q_x = 6x - By + 1 \quad \Rightarrow \quad A = 6 \quad B = 2 \)

2. \( f_x = 6xy - y^2 + y \quad \Rightarrow \quad f = 3x^2y - y^2x + yx + g(y) \)
   \( f_y = 3x^2 - 2yx + x + g'(y) = 3x^2 - 2xy + x \)
   \( g'(y) = 0 \quad \Rightarrow \quad f(x, y) = 3x^2y - y^2x + yx + \text{Const} \)

3. \( \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) + f(\frac{1}{2}, 2) = (3 - 1 + 1 - 3(\frac{1}{4})2 + 2 \cdot \frac{1}{2} - 1) \)
   \( = -3 - \frac{3}{2} + 2 - 1 = \frac{3}{2} \)

4. \( \int_C (6xy - y^2 + y)dx + (3x^2 - 3xy + x)dy = \int \mathbf{F} \cdot d\mathbf{r} - \int xy \ dy \)
   \( = \frac{3}{2} - \int_{y=1}^{2} dy = \frac{3}{2} - 1 = \frac{1}{2} \)
Problem 5 (8 points).
Say whether the following statements are true (T) or false (F). You may assume that all functions and vector fields have derivatives of all orders everywhere in their domain. You do not need to give reasons; this problem will be graded by answer only. You will receive +1 for correct answer, -1 for an incorrect answer, and 0 for no answer.

1. The divergence of $\nabla \times \vec{F}$ is zero, for every $\vec{F}$.
2. In a simply connected region, $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of $C$.
3. If $\nabla f = \vec{0}$, then $f$ is a constant function.
4. If $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is a constant vector field.
5. If $\vec{F} = P(x, y) \vec{i} + Q(x, y) \vec{j}$ satisfies $Q_y = P_x$, then $\vec{F}$ is conservative.
6. If the domain of a function $f$ is open and connected, then $\text{div}(\text{grad}(f)) = 0$.
7. If $\vec{F} = P(x, y) \vec{i} + Q(x, y) \vec{j}$, and $\vec{F}$ is conservative, then $Q_x - P_y = 0$.
8. If $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve $C$ in the domain of $\vec{F}$, then there is a function $f$ such that $\vec{F} = \text{grad}(f)$.

1. True
2. False: only if $\vec{F}$ is conservative
3. True
4. False
5. False: also need simply connected
6. False
7. True
8. True