Suppose that a particle of mass $m$ is acted on by a central force whose magnitude is proportional to $1/r^d$ for some integer $d$. That is

$$F = \frac{-mC}{r^{d+1}}r$$

where $r = |\mathbf{r}|$ and $C$ is a positive constant. In the case of a gravitational force, $d = 2$ and we know from Kepler’s 1st law that the orbits are ellipses with one of the foci at the origin.

Suppose instead that there is a circular orbit that passes through the origin. Find the unique value of $d$ that allows such an orbit.

Suggested method:

1. Show that $\mathbf{v} \times \mathbf{r}$ is a conserved quantity (angular momentum).

2. Let $\mathbf{r}_0$ be the point on the circle antipodal to the origin and let $\mathbf{r}_1$ be a point on the circle 90 degrees away from the origin and $\mathbf{r}_0$. Let $v_0$ and $v_1$ be the speed of the particle at $\mathbf{r}_0$ and $\mathbf{r}_1$ respectively. Compute the angular momentum at $\mathbf{r}_0$ and $\mathbf{r}_1$.

3. Consider the decomposition of the acceleration into tangential and normal components:

$$\mathbf{a} = v' \mathbf{T} + \kappa v^2 \mathbf{N}.$$

Take the inner product of both sides of this equation with $\mathbf{N}$ and evaluate at $\mathbf{r}_0$ and $\mathbf{r}_1$.

4. Use the three equations obtained in the previous steps to solve for $d$. 