Homework 3

Math 615

October 12, 2016
In this problem set, you will compute some Donaldson-Thomas invariants of “local $\mathbb{P}^1 \times \mathbb{P}^1$” and then use the GW/DT correspondence and the Gopakumar-Vafa conjecture to compute various Gromov-Witten invariants and Gopakumar-Vafa invariants.

Let $X = \text{Total}(K_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^1 \times \mathbb{P}^1)$ be the total space of the canonical line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. The cone of effective classes in $H_2(X, \mathbb{Z})$ is generated by the curves $C_1 = \mathbb{P}^1 \times \{\text{point}\}$ and $C_2 = \{\text{point}\} \times \mathbb{P}^1$ and we abbreviate the class $\beta = d_1[C_1] + d_2[C_2]$ by simply $(d_1, d_2)$.

It will be helpful in the below problems to recall from class the theorem that the value of the Behrend function at torus fixed points of the moduli space is $(-1)^{s(\beta) + n}$ for some function $s(\beta)$. The value $(-1)^{s(\beta)}$ can easily be determined in the cases we consider by finding some value of $n$ such that $I_n(X, \beta)$ is smooth.
Problem 1. Invariants in the class $\beta = (1, 0)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (1, 0)$:

$$Z_{(1,0)}^{DT}(X) = \sum_{n} e_{\text{vir}}(I_n(X, (1,0))) q^n.$$  

Recall from class that the generating function for counting boxes added onto a infinite bar with asymptotic shape \(\Box\) is given by

$$V_{\emptyset, \emptyset, \emptyset}(q) = V_{\emptyset, \emptyset, \emptyset}(q) \cdot \frac{1}{1 - q}$$

where

$$V_{\emptyset, \emptyset, \emptyset}(q) = M(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m}.$$  

2. Use the Gromov-Witten/Donaldson-Thomas correspondence to get a prediction for the Gromov-Witten potential in the class $(1, 0)$:

$$F_{(1,0)}^{GW}(\lambda) = \sum_{g} N_{g,(1,0)}^{GW}(X) \lambda^{2g-2}.$$  

What is the value of $N_{0,(1,0)}^{GW}$? What is the value of $N_{1,(1,0)}^{GW}$?

3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(1,0)}(X)$ for all $g$. 

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Problem 2. Invariants in the class $\beta = (1, 1)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (1, 1)$:

$$Z^\text{DT}_{(1,1)}(X) = \sum_n e_{\text{vir}}(I_n(X, (1, 1))) q^n.$$ 

The generating function for counting boxes added onto two infinite bars (along say the $x$ and $y$ axis), each with asymptotic shape $\sigma$, is given by

$$V_{\mathbf{D}, \mathbf{D}, \emptyset}(q) = V_{\emptyset, \emptyset, \emptyset}(q) \cdot \left(1 + \frac{q}{(1 - q)^2}\right).$$

2. Use the Gromov-Witten/Donaldson-Thomas correspondence to get a prediction for the Gromov-Witten potential in the class $(1, 1)$:

$$F^\text{GW}_{(1,1)}(\lambda) = \sum_g N^\text{GW}_{g,(1,1)}(X) \lambda^{2g-2}.$$ 

**Warning!** Beware that the GW/DT correspondence gives a prediction for $Z^\text{GW}_{(1,1)}(X)'$, the generating function for the possibly disconnected GW invariants. To get a prediction for $F^\text{GW}_{(1,0)}(X)$, you must pass between the disconnected and the connected invariants. This issue did not arise for the class $(1,0)$ because invariants in that class are automatically connected.

What is the value of $N^\text{GW}_{0,(1,1)}$? What is the value of $N^\text{GW}_{1,(1,1)}$?

3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(1,1)}(X)$ for all $g$. 


Problem 3. Invariants in the class $\beta = (2, 0)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (2, 0)$:

$$Z^\text{DT}_{(2,0)}(X) = \sum_n e_{\text{vir}}(I_n(X, (2, 0))) q^n.$$  

The generating function for counting boxes added onto an infinite bar with asymptotic shape $\mathfrak{B}$ is given by

$$V_{\emptyset, \emptyset, \emptyset}(q) = V_{\emptyset, \emptyset, \emptyset}(q) \cdot \frac{1}{(1 - q)(1 - q^2)}.$$  

2. Use the Gromov-Witten/Donaldson-Thomas correspondence to get a prediction for the Gromov-Witten potential in the class $(2, 0)$:

$$F^\text{GW}_{(2,0)}(\lambda) = \sum_g N^\text{GW}_{g,(2,0)}(X) \lambda^{2g-2}.$$  

**Warning!** You must pass from disconnected to connected invariants (see previous warning).

What is the value of $N^\text{GW}_{0,(2,0)}$? What is the value of $N^\text{GW}_{1,(2,0)}$?

3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(2,0)}(X)$ for all $g$. **Warning:** because the class $(2, 0)$ is divisible, using the Gopakumar-Vafa formula is a little trickier in this case.