Homework # 3

Due Thursday, October 13th

1. Let $X \subset \mathbb{P}^n$ be a projective variety and let $V \subset \mathbb{A}^m$ be an affine variety. Let $\phi : X \rightarrow Y$ be a morphism. Prove that $\phi$ is constant, i.e. there exists a $v \in V$ such that $\phi(x) = v$ for all $x \in X$.

2. Show that any irreducible conic in $\mathbb{A}^2$ is isomorphic to $Z(y - x^2)$ or $Z(xy - 1)$.

3. Let $X$ be an affine variety, and let $G$ be a finite group. Assume that $G$ acts on $X$, i.e. for every $g \in G$ we are given an automorphism $\phi_g : C \rightarrow C$ such that $\phi_{gh} = \phi_g \circ \phi_h$ for all $g, h \in G$ and if $e \in G$ is the identity, then $\phi_e$ is the identity morphism.

   (a) Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of $G$-invariant functions on $X$, i.e. all $f \in A(X)$ such that $f(p) = f(\phi_g(p))$ for all $g \in G$ and all $p \in X$. By a famous theorem of Emmy Noether, $A(X)^G$ is finitely generated, let $Y$ be the corresponding affine variety so that $A(Y) \cong A(X)^G$. The homomorphism $A(X)^G \rightarrow A(X)$ given by inclusion gives rise to a morphism of affine varieties $\pi : X \rightarrow Y$. Show that $Y$ can be considered as $X/G$, the quotient of $X$ by $G$ in the following sense:

   i. $\pi$ is surjective.

   ii. If $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there exists $g \in G$ such that $\phi_g(p) = q$.

   (b) For a given group action, is an affine variety with the above two properties uniquely determined?

   (c) Let $\mu_n = \exp \left( \frac{2\pi i k}{n} \right)$, $i = 0, \ldots, n - 1$ be the group of $n$th roots of 1. Let $\mu_n$ act on $\mathbb{A}^2$ by $\phi_{\zeta}(x, y) = (\zeta x, \zeta^{-1} y)$ for any $\zeta \in \mu_n$. Find a hypersurface $Y \subset \mathbb{A}^3$ such that $Y_n \cong \mathbb{A}^2 / \mu_n$. Show that $Y_n$ is not isomorphic to $\mathbb{A}^2$. (Hint: you may use a topological argument).
4. Let \( C \subset \mathbb{P}^2 \) be the cubic curve given by \( zy^2 = x(x^2 - z^2) \). In class we defined an automorphism \( \phi : C \to C \) which was given by \( \phi(a, b) = (-1/a, -b/a^2) \) in the affine coordinates \( a = x/z, b = y/z \), and the two points \((0 : 0 : 1), (0 : 1 : 0)\) where the formula is not well defined are exchanged by \( \phi \). Note that \( \phi \circ \phi \) is the identity, so \( \phi \) defines an action of \( G = \mathbb{Z}/2\mathbb{Z} \) on \( C \).

(a) Determine all the fixed points of \( \phi \).

(b) Show that there is a morphism \( \pi : C \to \mathbb{P}^1 \) such that \( \pi(p) = \pi(q) \) if and only if \( p = q \) or \( \phi(p) = q \). In other words, show that \( C/G \cong \mathbb{P}^1 \).

(c) Consider the variant of the above automorphism given by \( \phi'(a, b) = (-1/a, b/a^2) \) (with \( \phi' \) also exchanging \((0 : 0 : 1)\) and \((0 : 1 : 0)\)). Determine all the fixed points of \( \phi' \).

(d) Show that the quotient \( C/G \) where the action of \( G \) on \( C \) is given by \( \phi' \) is isomorphic to the original curve \( C \subset \mathbb{P}^2 \). (Hint: consider the action on the affine patch of \( C \) where \( x \neq 0 \) and use problem 3a). I will remark that the fact that \( C/G \cong C \) is unusual and special to this particular elliptic curve.