Suppose we have two dependent variables $x(t)$ and $y(t)$. We can write a system of equations for these variables,

$$
\begin{align*}
  x'(t) &= f(t, x, y) \\
y'(t) &= g(t, x, y)
\end{align*}
$$

Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

and $F(t, \vec{x}(t)) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}$

Then we can write (1) as

$$\vec{x}'(t) = F(t, \vec{x}(t))$$
The system is autonomous if

\[ \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \]

That is, the system does not depend on \( t \) explicitly.

**Example:** Given the graph

\[ \begin{array}{c}
\dot{\mathbf{x}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \text{ position vector.} \\
\dot{\mathbf{x}}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \text{ velocity vector.} \\
\end{array} \]

So we can write an ODE system

\[ \dot{\mathbf{x}}(t) = \mathbf{F}(t; \mathbf{x}(t)) \]
Suppose

\[ f(t, x, y) = a(t) x(t) + b(t) y(t) + g_1(t) \]
\[ g(t, x, y) = c(t) x(t) + d(t) y(t) + g_2(t) \]

Then the system in (1) becomes

\[
\begin{align*}
    x'(t) &= a(t) x(t) + b(t) y(t) + g_1(t) \\
y'(t) &= c(t) x(t) + d(t) y(t) + g_2(t)
\end{align*}
\]  \tag{2}

which is a linear system of ODEs.

Let

\[
A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}
\]

We can write (2) as

\[
\dot{\mathbf{x}}(t) = A(t) \mathbf{x}(t) + \mathbf{g}(t)
\]  \tag{3}

where \( \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) and \( \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \)
If the matrix $A$ is independent of $t$, i.e., $A$ is a constant matrix. Then the we have a constant coefficient linear system.

The vector function $\vec{G}(t)$ is called the forcing function.

If $\vec{G}(t) = \vec{0}$, then (3) becomes

$$\dot{x}'(t) = A(t) \dot{x}(t)$$

and the system is called an homogeneous system, otherwise (3) is non homogeneous.
Consider the matrix $A$. If there exists a vector $\vec{v}$ such that

$$A\vec{v} = \vec{0}, \quad \text{then} \quad \vec{v} \in \text{Null}(A)$$

$\Rightarrow$ Null$(A) = \{ \vec{v} \mid A\vec{v} = \vec{0} \}$ (Null space of $A$)

If $\vec{v} \in \text{Null}(A)$ and $\lambda$ is a scalar, then $\lambda \vec{v} \in \text{Null}(A)$

Let $\vec{x}_p$ be a particular solution to a linear algebraic system of equations, $A\vec{x} = \vec{b}$

Then $A\vec{x}_p = \vec{b}$

Let $\vec{v} \in \text{Null}(A)$ and $\lambda$ be a scalar

$$A(\vec{x}_p + \lambda \vec{v}) = A\vec{x}_p + A(\lambda \vec{v}) = \vec{0}$$

$$= A\vec{x}_p = \vec{b}$$

$$A(\vec{x}_p + \lambda \vec{v}) = \vec{b}$$
\[ \dot{\bar{Y}}_{P} + \bar{L} \bar{V} \] is the general solution of the algebraic system.

Now, consider the linear ODE
\[ \frac{d}{dt} y(t) + p(t) y(t) = g(t) \tag{4} \]

Define the operator \( L \) (Linear operator)
\[ L = \frac{\dot{y}}{dt} + p(t) \]

So that
\[ L[y(t)] = \frac{dy(t)}{dt} + p(t) y(t) \]

\[ \therefore \tag{4} \] can be written as
\[ L[y] = g(t) \tag{5} \]

Similar to system of algebraic equations, the general solution of \( \tag{5} \) is given by
\[ y_p(t) + C v(t) \]

where \( C \) is a constant.
$y_p(t)$ is one particular solution to
\[ L[y] = g(t) \]
and
\[ v(t) \text{ is a solution to } L[v(t)] = 0 \]
Since \( \text{Null}(L) = \{ v(t) \mid L[v(t)] = 0 \} \)
we can say that \( v(t) \in \text{Null}(L) \).

\underline{properties of operator L}

1. \( L(f(t) + g(t)) = L(f(t)) + L(g(t)) \)
2. \( L(cf(t)) = cL(f(t)) \)
3. \( L(0) = 0 \)
In general, given a nonhomogeneous ODE
\[ \frac{dy}{dt} + p(t) y(t) = g(t) \]

or \[ L[y(t)] = g(t) \]

let \( y_H(t) \) be the solution to the homogeneous problem
\[ L[y_H(t)] = 0 \]

and \( y_p(t) \) be a solution to the nonhomogeneous problem
\[ L[y_p(t)] = g(t) \]

The general solution of the ODE is
\[ y(t) = c y_H(t) + y_p(t) \]

where \( c \) is a constant
Example: Consider

\[
\frac{dy}{dt} + \frac{2y}{t} = \frac{\sin(t)}{t^2}
\]  (from q. 3.1)

with general solution

\[
y(t) = \frac{c}{t^2} - \frac{\cos(t)}{t^2}
\]

Consider the homogeneous problem

\[
\frac{dy}{dt} + \frac{2y}{t} = 0
\]

\[
\frac{dy}{dt} = -\frac{2y}{t}
\]

\[
\int \frac{dy}{y} = \int \frac{-2}{t} \, dt + C_1
\]

\[
\ln y = -2 \ln t + C_1
\]

\[
y(t) = e^{\ln\left(\frac{1}{t^2}\right)} + C_1 = \frac{C_2}{t^2}
\]

\[
\Rightarrow y_H(t) = \frac{C_2}{t^2}
\]

\[
\therefore \text{ from } (\star), \quad y_p(t) = -\frac{\cos(t)}{t^2}
\]

\[
\Rightarrow y(t) = y_H(t) + y_p(t) = \frac{C_2}{t^2} - \frac{\cos(t)}{t^2}
\]