Last class
We started with solving
\[ \dot{q}(t+1) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} q(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t \]

Using the variation of parameter formula
\[ \dot{q}(t) = \int_0^t \Phi^{-1}(s) \dot{g}(s) ds + \Phi(t) c \]

We got
\[ \Phi = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \]

and
\[ \int_0^t \Phi^{-1}(s) d_s = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t \]

The general solution of the system is
\[ q(t) = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t \]
\[ y(t) = C_1 \left( \frac{1}{2} \right) e^{3t} + C_2 \left( \frac{-1}{2} \right) e^{-t} + \left( \frac{1}{2} \right) e^t \]

Observe that this solution is the same as what we got using the method of undetermined coefficients.
Second order constant coefficient ODEs

Consider

\[ y'' + by' + cy = 0 \], Homogeneous (unforced)

\[ y'' + by' + cy = g(t) \], Non homogeneous (forced)

where \( g(t) \) is the forcing function.

Vibrating Springs (Mass - spring)

From Hooke's law,

restoring force \( = -kx \)

where \( k > 0 \) is the spring constant.
From Newton's second law of motion,

\[ \text{Sum of forces} = m a \]

\[ a = \frac{d^2 x}{dt^2} \]

\[ -kx = m \frac{d^2 x}{dt^2} \]

- We have ignored external resisting forces.

\[ \frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \]

(undamped) simple harmonic motion

Let us introduce damping:

\[ \text{damping force} = -\mu V = -\mu \frac{dx}{dt} \]

where \( \mu > 0 \) damping coefficient.
Using Newton's 2nd law,

\[-kx - \mu \frac{dx}{dt} = m \frac{d^2x}{dt^2}\]

\[\frac{d^2x}{dt^2} + \frac{\gamma}{m} \frac{dx}{dt} + \frac{k}{m} x = 0\]

Let \( \gamma = \frac{\mu}{m} \)

\[\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{k}{m} x = 0\]

damping term.
2) **LCR Circuit**

- L - inductor, C - capacitor, and R - resistor

![Diagram of an LCR circuit](image)

Let I - current, \( q \) - charge on capacitor

Voltage across the circuit:

- **resistor**: \( IR \)
- **inductor**: \( L \frac{dI}{dt} \)
- **capacitor**: \( \frac{q}{C} \)

**Kirchhoff's 2nd Law**

Sum of voltage drops = sum of applied voltage
\[ L \frac{dI}{dt} + IR + \frac{\Phi}{C} = V(t) \]

But \[ I = \frac{d\Phi}{dt} \]

\[ L \frac{d^2\Phi}{dt^2} + R \frac{d\Phi}{dt} + \frac{\Phi}{C} = V(t) \]

Damping term for driving function

3) Linear pendulum

Consider the force in the direction perpendicular to the pendulum:

\[ \sin \theta = \frac{f}{mg} \]
\[ f = -mg \sin \Theta \]

acceleration, \; velocity,

\[ v \propto \frac{dv(t)}{dt} \]

\[ v = L \frac{d\theta}{dt} \]

\[ \Rightarrow a = L \frac{d^2\theta}{dt^2} \]

Now, using Newton's second law.

\[ f = ma \]

\[ -mg \sin \Theta = mL \frac{d^2\theta}{dt^2} \]

\[ \frac{d^2\theta}{dt^2} \theta + g \frac{\sin \Theta}{L} = 0 \]

Suppose we are interested in small \( \Theta \), then we can use the approximation \( \sin \Theta \approx \Theta \) for small \( \Theta \).
\[ O'' + g \cdot \frac{\theta}{L} = 0 \]  

ODE for a simple linear pendulum.

Now, consider the homogeneous equation

\[ y'' + by' + cy = 0 \]

Let us write this equation as a system of first order ODEs.

Let

\[ y_1 = y \]
\[ y_2 = y' \]
\[ y_1' = y_1 = y_2 \]
\[ y_2' = y'' \]

From (i), \[ y'' = -by' - cy \]
\[ y'' = -by_2 - cy_1 \]
\[ \begin{align*}
y_1' &= y_2 \\
y_2' &= -by_2 - cy_1,
\end{align*} \]

\[
\begin{pmatrix}
y_1' \\
y_2'
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-c & -b
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}.
\]

Following the same procedure, we can write higher order equations as a system of 1st order ODEs.

**Question:** How do we solve the 2nd order ODE?

\[
y'' + by' + cy = 0 \quad \text{(1)}
\]

Guess: \( y(t) = e^{\lambda t}, \quad y' = \lambda e^{\lambda t} \)

Put \( y \) into (1): \( y'' = \lambda^2 e^{\lambda t} \)

\[
\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0
\]

\[
(\lambda^2 + b\lambda + c) e^{\lambda t} = 0.
\]
\( e^{at} \neq 0. \)

\[ \Rightarrow \lambda^2 + 6\lambda + c = 0 \]

Using the quadratic formula,

\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \]

**Forms of Solution**

1. \( \lambda_1 \) and \( \lambda_2 \) are real and distinct.

\[ y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \]

2. \( \lambda_1 = \lambda_2 = \lambda \) (repeated roots)

\[ y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \]

3. \( \lambda \) is complex, let \( \lambda = \alpha \pm i\beta \).

\[ y(t) = c \left( e^{\alpha t} (e^{i\beta t}) + e^{\alpha t} (e^{-i\beta t}) \right) = e^t \left( c_1 \cos(\beta t) + c_1 \sin(\beta t) + c_2 \cos(\beta t) - i c_2 \sin(\beta t) \right) \]
\[ y(t) = e^{\lambda t} \left[ (C_1 + C_2) \cos(\beta t) + i(C_1 - C_2) \sin(\beta t) \right] \]

Let \( k_1 = C_1 + C_2 \) and \( k_2 = i(C_1 - C_2) \)

\[ y(t) = e^{\lambda t} \left[ k_1 \cos(\beta t) + k_2 \sin(\beta t) \right] \]

\[ y(t) = k_1 e^{\lambda t} \cos(\beta t) + k_2 e^{\lambda t} \sin(\beta t) \]