Errors in Numerical Approximations

Two main types of error

* Global (accumulated) error

  This is as a result of

  i) using an approximation formula
  ii) the input $y_n$ at each step is an approximation.

Suppose we use the analytic solution $\phi(t)$ at the $n$th step as input. Then we have only i) left which gives us the Local Truncation error (L.T.E.).

* L.T.E is the error that arises as a result of using an approximation formula.
Local Truncation Error (L.T.E.)

The L.T.E. error at the $(n+1)$th step is given by

$$e_{n+1} \triangleq \varphi(t_{n+1}) - y_{n+1} \quad (1)$$

$$\varphi(t_{n+1}) = \varphi(t_n + \theta) \quad (2)$$

Taylor expand

$$\varphi(t_n + \theta) = \varphi(t_n) + \theta \varphi'(t_n) + \frac{\theta^2}{2} \varphi''(t_n) + \cdots$$

From Euler's formula,

$$y_{n+1} = y_n + \theta f(t_n, y_n) \quad (3)$$

Sub. (2) and (3) into (1).

$$e_{n+1} = (\varphi(t_n) - y_n) + \theta (\varphi'(t_n) - f(t_n, y_n)) + \frac{\theta^2}{2} \varphi''(t_n) + \cdots$$
But \( \phi(t_n) \approx y_n \) and \( \phi'(t_n) = f(t_n, y_n) \) for very small \( h \).

\[
\therefore \quad \epsilon_{n+1} \approx \frac{h^2}{2} \phi''(t_n) + \cdots
\]

\[
\epsilon_{n+1} = h^2 x (\text{some constant}) \quad \text{as} \quad h \to 0.
\]

\[
\Rightarrow \quad \text{for very small} \ h, \quad \text{the L.T.E. is proportional to scales as} \ h^2.
\]

Suppose we want to solve our IVP on \([0, T]\) with step-size \( h \).

\[
\Rightarrow \quad \text{the number steps from} \ 0 \ \text{to} \ T \ \text{is} \quad N = \frac{T}{h}
\]

and at each step we make an error of \( \epsilon_{n+1} = h^2 x (\text{some constant}) \).

\[
\therefore \quad \text{the global error at the} \ T \ \text{step is}
\]

\[
\frac{T}{h} \times h^2 x (\text{some constant}) = h x (\text{new constant}) \quad \text{as} \quad h \to 0
\]
This implies that the global error scales as $h$ for small $h$.

**Improved Euler method**

For Euler's method, we use the slope at $t_n$ to compute $y_{n+1}$. Let's try using the average of the slopes at $t_n$ and $t_{n+1}$, that is, we use

$$f(t_n, y_n) + f(t_{n+1}, y_{n+1})$$

\[ (\ast) \]

Sub. (\ast) into Euler's formula.

$$y_{n+1} = y_n + h \left[ \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} \right] \quad (\ast\ast)$$
what if $f(t, y)$ is something like
$f = 2t + y$?

we have our unknown as input. What if we compute $y_{n+1}$ using Euler's formula?

$$y_{n+1} = y_n + h f(t_n, y_n)$$

Then put in (1):

$$y_{n+1} = y_n + \frac{h}{2} \left[ f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1}) \right]$$

This method is called the Improved Euler's method

$$y_{n+1} = y_n + \frac{h}{2} \left[ f(t_n, y_n) + f(t_{n+1}, y_n + h f(t_n, y_n)) \right]$$

$n = 0, 1, 2, \ldots$
We can write this as a two-stage step-method that is,

\[ \tilde{y}_{n+1} = y_n + h f(t_n, y_n) \]

\[ y_{n+1} = y_n + \frac{h}{2} \left[ f(t_n, y_n) + f(t_n + \frac{h}{2}, \tilde{y}_{n+1}) \right] \]

For the same error analysis as we did for Euler's method, we can show that the local truncation error scales as \( h^3 \) as \( h \to 0 \) and global error scales as \( h^2 \) as \( h \to 0 \).

* This type of multi-stage methods are called Runge-Kutta methods (R-K).

* ode 45 in matlab uses 4th order R-K.
Examples: Use improved Euler's method to approximate the numerical solution of the IVP

\[ y' = 1 - t + 4y, \quad y(0) = 1 \]
on \[ [0, 1] \]
with \( h = 0.1 \).

Improved Euler method is given by

\[ \dot{Y}_{n+1} = Y_n + h f(t_n, Y_n) \]

\[ Y_{n+1} = Y_n + \frac{h}{2} \left( f(t_n, Y_n) + f(t_{n+1}, \dot{Y}_{n+1}) \right) \]

\( h = 0.1 \), \( t_0 = 0 \), \( Y_0 = 1 \), \( f(t, y) = 1 - t + 4y \).

when \( n = 0 \),

\[ \dot{Y}_1 = Y_0 + h f(t_0, Y_0) = 1 + 0.1 \left( 1 - 0 + 4(1) \right) \]

\[ = 1.5 \]

\[ Y_1 = Y_0 + \frac{0.1}{2} \left( 5 + (1 - 0.1 + 4(1.5)) \right) \]

\[ = 1.595 \]