**Order** of an ODE is the highest number of derivatives in the equation.

Examples:

(i) \[ y'' - 4y' + 4y = 0 \] \(- 2\text{nd order}\)

(ii) \[ y' + Ty''' = \sin(t) \] \(- 4\text{th order}\)

**Linear / Nonlinear ODEs**

If an ODE contains only linear functions of \( y, y', y'' \), then it is linear.

More precisely,

\[ a(t) \frac{dy}{dt} + b(t) y = c(t) \] \(- 1\text{st order linear}\)

\[ a(t) \frac{d^2y}{dt^2} + b(t) \frac{dy}{dt} + c(t) y = d(t) \] \(- 2\text{nd order linear}\)
Example:

1. \( x^2 y'' + xy' + y = 0 \) — Linear
2. \( yy' = 1 \) — Nonlinear
3. \( t^2 y'' - \sin(xy)' = e^t \) — Linear
4. \( y'' + \sin(y) = 0 \) — Nonlinear

**Linear ODE:** — Usually have solution
- There are techniques for solving them exactly/analytically.

**Nonlinear ODE:** — May not have solution
- Or have unique solution.
- Or we may not be able to solve analytically.
- Usually they are solved numerically or by approximation.
Example: Solve the IVP (Initial Value Problem)

\[
\frac{dy}{dt} = e^{-2t}, \quad y(0) = 5, \quad t > 0
\]

\[
\frac{dy}{dt} = e^{-2t} \quad \Rightarrow \quad \int dy = \int e^{-2t} \, dt
\]

\[
y = \frac{e^{-2t}}{-2} + C \quad \text{--- general solution}
\]

Apply the initial condition \( y(0) = 5 \)

\[
5 = \frac{e^{-2(0)}}{-2} + C \quad \Rightarrow \quad C = 5 + \frac{1}{2} = \frac{11}{2}
\]

\[
\therefore \quad y = \frac{-1}{2} e^{-2t} + \frac{11}{2}
\]
Let us solve the same problem in a slightly different way (using definite integral). We have

\[ \frac{dy}{dt} = e^{-2t} \]

\[ \int_{y_0}^{y} \, dz = \int_{t_0}^{t} e^{-2u} \, du \]

From I.C. \( y(0) = 5 \Rightarrow t_0 = 0, \, y_0 = 5 \)

\[ \int_{5}^{y} \, dz = \int_{0}^{t} e^{-2u} \, du \]

\[ z \bigg|_{5}^{y} = -\frac{1}{2} e^{-2u} \bigg|_{0}^{t} \]

\[ y - 5 = \frac{1}{2} e^{-2t} + \frac{1}{2} \]

\[ y = \frac{11}{2} - \frac{1}{2} e^{-2t} \]
Example: Solve the IVP

\[ \frac{dy}{dt} = e^{-t^2}, \quad y(0) = \frac{1}{2} \]

\[ \frac{dy}{dt} = e^{-t^2} \]

\[ \int y' \, dt = \int e^{-t^2} \, dt \]

\[ \int_{y_0}^{y} \, dt = \int_{t_0}^{t} e^{-u^2} \, du \]

\[ y_0 = \frac{1}{2} \quad t_0 = 0 \]

\[ \int_{1/2}^{y} 1 \, du = \int_{1/2}^{t} e^{-u^2} \, du \]

\[ z = \int_{1/2}^{y} e^{-u^2} \, du \]

\[ y = \int_{0}^{t} e^{-u^2} \, du + \frac{1}{2} \]
The integral on the RHS does not have a closed form. Therefore, we can write the solution of our equation as a definite integral.

**Existence & Uniqueness of Solutions of ODEs**

Given an I.V.P

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

Fundamental questions to ask:

1. Does a solution exist?
2. If a solution exists, is the solution unique?
Example: Solve

\[ xy' = 4, \quad y(0) = 5 \]

Suggestion try to solve this --

Solution: \[ y = 4 \ln |x| + C \]

But \[ \ln (0) = -\infty \]

\[ \Rightarrow \] the IVP does not have a solution that satisfies the initial condition

\[ \Rightarrow \] the equation does not have a smooth (continuous) solution near \( x = 0 \).
Example: Solve
\[
\frac{dy}{dx} = 3y^{2/3}, \quad y(1) = 0
\]
\[
\int y^{-2/3} \, dy = \int 3 \, dx + C_1
\]
3 \( y^{1/3} \) = 3 \( x + C_1 \)
\[
y^{1/3} = x + C_2 \quad (C_2 = C_1/3)
\]
y(2) = 0 \Rightarrow 0 = 2 + C_2
\[
C_2 = -2
\]
y^{1/3} = x - 2

Observe that if \( y = 0 \),
\[
\frac{dy}{dx} = 0, \quad y(2) = 0
\]
\( y = 0 \) is also a solution of the IVP
\[
\Rightarrow \quad \text{The IVP does not have a unique solution close to (2, 0)}.
\]
Picard's theorem (existence & uniqueness of solutions of ODEs)

Given an IVP

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

If \( f(x, y) \) is continuous and \( \frac{\partial f}{\partial y} \) exists and is continuous near some point \((x_0, y_0)\), then \( \exists \) (there exist) a solution to the IVP near \((x_0, y_0)\) and the solution is unique near this point.

Check:

i. \( xy' = 4 \), \( y(0) = 5 \)

\[ y' = \frac{4}{x} \]

\[ f(x, y) = \frac{4}{x} \]

At \( x = 0 \), \( f(x, y) \) is not defined. \( \therefore \) This violates the assumptions of the theorem.
\( \frac{dy}{dx} = 3y^{2/3} \)

\[ f(x,y) = 3y^{2/3} \quad \text{is continuous} \]

\[ \frac{df}{dy} = 2y^{-1/3} = \frac{2}{y^{1/3}} \]

At \( y = 0 \), \( \frac{df}{dy} \) does not exist.

This violates the assumptions of the theorem.

\underline{Note:} that \( f(x,y) \) is not continuous or \( \frac{df}{dy} \) is not continuous does not necessarily mean that an IVP does not have a solution or a solution is not unique. Also, the statement of the theorem is not if and only if.