Name:          SID:
Instructor:    Section:

Instructions

• The total time allowed is 50 minutes.
• The total score is 40 points.
• Use the reverse side of each page if you need extra space.
• Show all your work. A correct answer without intermediate steps will receive no credit.
• Calculators, phones and cheat sheets are not allowed.

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</table>
1. (5 points) Solve the following initial value problems for $y(t)$:

$$ty' + 2y = \cos(t), \quad y(\pi) = 0.$$
2. (10 points) Consider the following system of first order ODEs

\[
\frac{dy_1}{dt} = 2y_1(t) - 2y_2(t) \\
\frac{dy_2}{dt} = 2y_1(t) + 2y_2(t)
\]

(a) Find the general solution of the system. Convert any complex exponentials in your solutions into “real forms” involving sines and cosines.

Solution:

(b) Use the initial conditions \(y_1(0) = 1\) and \(y_2(0) = 2\) to find the constants in your solution.

Solution:
3. (7 points) Determine the value of $k$ for which the following equation is exact

$$(y \cos(x) + kxe^y) \, dx + (\sin(x) + x^2e^y - 1) \, dy = 0.$$ 

Solution:

Use this value of $k$ together with the initial condition $y(\pi) = 0$ to solve the equation.
4. (6 points) Suppose the system of equations $\dot{\vec{Y}}(t) = A\vec{Y}(t)$ has the vector field

(i) This vector field suggests that the eigenvalues associated with the system are;

(a) Distinct real  
(b) Repeated real  
(c) Complex  
(d) None of the above

(ii) Given the initial condition $y_1(0) = 3$ and $y_2(0) = 2$, use the vector field to determine the value of $\vec{Y}(t)$ as $t \rightarrow \infty$?

Solution:
(iii) Which of the following Matlab commands will produce this direction field?

(a) 
```matlab
[Y1,Y2] = meshgrid([-4:0.5:4,-4:0.5:4]); % creates a meshgrid
U1 = 1*Y1 + 1*Y2; % the first equation in the system
U2 = 1*Y1 + 1*Y2; % the second equation in the system
quiver(Y1,Y2,U1,U2) % create the vector field
figure()
quiver(Y1,Y2,U1,U2,2) % create the vector field with longer arrows
title('Vector Field')
xlabel('y_1'), ylabel('y_2')
```

(b) 
```matlab
[Y1,Y2] = meshgrid([-4:0.5:4,-4:0.5:4]); % creates a meshgrid
U1 = -1*Y1 - 1*Y2; % the first equation in the system
U2 = 2*Y1 - 1*Y2; % the second equation in the system
quiver(Y1,Y2,U1,U2) % create the vector field
figure()
quiver(Y1,Y2,U1,U2,2) % create the vector field with longer arrows
title('Vector Field')
xlabel('y_1'), ylabel('y_2')
```

(c) 
```matlab
[Y1,Y2] = meshgrid([-4:0.5:4,-4:0.5:4]); % creates a meshgrid
U1 = -2*Y1 + 1*Y2; % the first equation in the system
U2 = 1*Y1 - 2*Y2; % the second equation in the system
quiver(Y1,Y2,U1,U2) % create the vector field
figure()
quiver(Y1,Y2,U1,U2,2) % create the vector field with longer arrows
title('Vector Field')
xlabel('y_1'), ylabel('y_2')
```

(d) 
```matlab
[Y1,Y2] = meshgrid([-4:0.5:4,-4:0.5:4]); % creates a meshgrid
U1 = 1*Y1 + 2*Y2; % the first equation in the system
U2 = 3*Y1 + 2*Y2; % the second equation in the system
quiver(Y1,Y2,U1,U2) % create the vector field
figure()
quiver(Y1,Y2,U1,U2,2) % create the vector field with longer arrows
title('Vector Field')
xlabel('y_1'), ylabel('y_2')
```
5. (12 points) Consider the following ODE

\[
\frac{dy}{dt} = \lambda(y^2 - 4), \quad \text{where } \lambda > 0.
\]

(a) Find all the equilibria (steady state solutions) of the differential equation.

Solution:

(b) Sketch the graph of \(\frac{dy}{dt}\) vs \(y(t)\) and use it to determine which of these equilibria is stable, unstable, or semi-stable.

Solution:
(c) Use the initial condition $y(0) = 1$ to find a solution to the equation. Hint: You may need the partial fraction
\[
\frac{1}{(y-a)(y-b)} = \frac{A}{(y-a)} + \frac{B}{(y-b)}.
\]

Solution:

(d) Find the limit of this solution $y(t)$ as $t \to \infty$?

Solution: