Constant Coefficient Linear Systems

Consider the following system of ODEs.

\[ \begin{align*}
    y_1' &= y_1 + 2y_2 \\
    y_2' &= 3y_1 + 2y_2
    \end{align*} \quad (1)
\]

Let \( \vec{y}(t) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \), \( \vec{y}'(t) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \)

We can write (1) as

\[ \vec{y}'(t) = A \vec{y}(t) \quad (2) \]

where

\[ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \]

**Q1:** How can we solve this system for \( \vec{y}(t) \)?

Let us consider the scalar ODE,

\[ \frac{dy}{dt} = \lambda y(t) \quad (3) \]

\( \lambda \) - growth rate
The solution of this equation is
\[ y_{eq} = Ce^{\lambda t} \]

Observe that (2) and (3) are first order linear. We expect their solutions to have a similar form.

Take
\[ \vec{y}(t) = CE^{\lambda t} \vec{v} \quad (4) \]

where \( \lambda \) is a scalar (growth rate)
\( C \) - constant
\( \vec{v} \) - constant vector

Put the solution in (4) into the system
\[ \dot{\vec{y}}(t) = \Lambda \vec{y}(t) \]
\[ \lambda CE^{\lambda t} \vec{v} = \Lambda CE^{\lambda t} \vec{v} \]
\[ \lambda \vec{v} = \Lambda \vec{v} \]
\[ \Lambda \vec{v} = \lambda \vec{v} \quad (\text{eigenvalue problem}) \]
This implies that $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $\vec{v}$.

\[ A\vec{v} = \lambda \vec{v} \]

\[ \Rightarrow A\vec{v} - \lambda \vec{v} = \vec{0} \]

\[ (A - \lambda I)\vec{v} = \vec{0} \quad \text{(System of algebraic equations)} \]

We seek a non-trivial $\vec{v}$ (i.e. $\vec{v} \neq \vec{0}$) that satisfies this equation.

\[ \Rightarrow \det(A - \lambda I) = 0 \quad \text{must hold.} \]

To get the eigenvalue of matrix $A$, we solve

\[ \det(A - \lambda I) = 0 \quad \text{(Characteristic equation)} \]

or

\[ |A - \lambda I| = 0 \]
And for the corresponding eigenvalue, we solve

\[ (A - \lambda I) \vec{v} = \vec{0} \]

Thus, \[ \vec{y}(t + 1) = c e^{\lambda t} \vec{v} \]
is a solution of the system.

If \[ \vec{y}_1(t) = c_1 e^{\lambda t} \vec{v}_1 \]
and \[ \vec{y}_2(t) = c_2 e^{\lambda t} \vec{v}_2 \]
then are solutions \[ \vec{y}_1(t) + \vec{y}_2(t) \]
of a linear system, then

\[ \vec{y}(t) + \vec{y}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2 \]
is also a solution of the system.

This is called principle of superposition.
Let us return to the example
\[ \dot{y}(t) = A \cdot y(t) \]
where
\[ A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \]

Let \( \lambda \) be an eigenvalue of \( A \),
then \[ |A - \lambda I| = 0 \]

\[ \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = 0 \]

\[ (1-\lambda)(2-\lambda) - 6 = 0 \]
\[ \lambda^2 - 3\lambda - 4 = 0 \]
\[ \lambda_1 = 4 \quad \text{or} \quad \lambda_2 = -1 \]