Example: Solve

$$(3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0 \quad \text{(1)}$$

$M(x,y) = 3xy + y^2$, $My = 3x + 2y$

$N(x,y) = x^2 + xy$, $Nx = 2x + y$

Let check if $F$ an I.F. that is a function of $x$ only

$$\frac{My - Nx}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x+y)} = \frac{1}{x}$$

To get the I.F., we solve the ODE

$$\frac{1}{h} = \frac{1}{x} \cdot h$$

$$\int \frac{1}{h} \, dh = \int \frac{1}{x} \, dx + C$$
\[ \ln(h) = \ln(x) + C_1, \]
\[ h(x) = C_2 e^{\ln(x)} = C_2 x \]

Set \( C_2 = 1 \)

\[ h(x) = x \]

Multiply through (1) by \( h \).

\[ (3x^2y + xy^2) + (x^3 + x^2y) \frac{dy}{dx} = 0 \]

\[ M = 3x^2y + xy^2 \]
\[ N = x^3 + x^2y \]
\[ N_x = 3x^2 + 2xy \]

Check if equation is exact.

\[ M_y = 3x^2 + 2xy = N_x \]

Equation is exact.
We want to find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M, \quad \frac{\partial \psi}{\partial y} = N$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = 3x^2y + xy^2$$

$$\psi(x,y) = x^3y + \frac{x^2y^2}{2} + \psi_1(y)$$

$$\frac{\partial \psi}{\partial y} = x^3 + xy^2$$

$$\Rightarrow \psi(x,y) = x^3y + \frac{x^2y^2}{2} + \psi_2(x)$$

Comparing \(\psi_1\) and \(\psi_2\), we can set

$$\psi_1(y) = \psi_2(x) = 0.$$}

$$\psi(x,y) = x^3y + \frac{x^2y^2}{2}$$

Our solution is

$$x^3y + \frac{x^2y^2}{2} = C.$$
Numerical Approximation of Solutions of ODEs.

Euler's method (Tangent line method)

Given an IVP

\[ y' = f(t, y), \quad y(t_0) = y_0 \]  \hspace{1cm} (1)

Suppose the analytic solution is

\[ y = \phi(t) \]

tangent line

\[ \frac{y - y_0}{y_1 - y_0} = \frac{t - t_0}{t_1 - t_0} \]
The equation of the tangent line is

\[(y - y_0) = \left. \frac{dy}{dt} \right|_{t_0} (t - t_0)\]

\[\left. \frac{dy}{dt} \right|_{t_0} = f(t_0, y_0)\]

\[(y - y_0) = f(t_0, y_0) (t - t_0)\]

\[y = y_0 + f(t_0, y_0) (t - t_0)\]

Therefore,

goal: use the tangent lines to find an approximate solution to the IVP.

To get \(y_1\) at point \(t_1\),

\[y_1 = y_0 + f(t_0, y_0) (t_1 - t_0)\]

and for \(y_2\) at \(t_2\),

\[y_2 = y_1 + f(t_1, y_1) (t_2 - t_1)\]
Continuing this way, we have

\[ y_{n+1} = y_n + f(t_n, y_n) (t_{n+1} - t_n) \]

for the \((n+1)\)th approximation.

Assuming the step size in time is uniform

say \( t_{n+1} - t_n = h \)

\( t_{n+1} = h + t_n \)

Then the Euler's method is given by

\[ y_{n+1} = y_n + h \frac{d}{dt} \big|_{t_n} \]

\( n = 0, 1, 2, \ldots \)
Example: Solve \( y' = (2-t)y \), \( y(0) = 1 \) for \( 0 \leq t \leq 1 \)

First, partition \([0, 1]\) into subintervals

\[
0 \to t_0 \to t_1 \to \cdots \to t_n
\]

Take \( h = 0.1 \),
\( t_0 = 0 \), \( y_0 = 1 \), \( f(t, y) = (2-t)y \)

Euler's method

\[
y_{n+1} = y_n + h f(t_n, y_n)
\]

When \( n = 0 \), we have

\[
y_1 = y_0 + h f(t_0, y_0)
\]

\[
y_1 = 1 + 0.1 \left[ (2-0)(1) \right] = 1.2
\]

When \( n = 1/10 \)

\[
y_2 = y_1 + h f(t_1, y_1)
\]

\[
y_2 = 1.2 + 0.1 \left[ (2-0.1)(1.2) \right] = 1.428
\]
\[ y_3 = y_2 + h \cdot f(t_2, y_2) \]
\[ = 1.428 + 0.1 \left[ (2 - 0.2) \cdot 1.428 \right] \]
\[ = 1.6850 \]