This midterm exam has 5 questions on 7 pages, for a total of 45 marks.

Duration: 50 minutes

Full Name (Last, First, All middle names): ____________________________

Student-No: ______________________ Course Section: _____________

Signature: _______________________________________________________

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Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- Unless otherwise stated, you should give complete arguments and explanations for all your answers and calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
1. (a) (3 marks) Write the contrapositive of the conditional statement: If \( x^2 > y^2 \), then \( x > y \) or \( x < -y \).

**Solution:** If \( x \leq y \) and \( x \geq -y \), then \( x^2 \leq y^2 \).

(b) (4 marks) Determine whether the following statement is true or false — Put True or False in the box, and **Justify your answers**.

\[ \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \text{ such that } y \neq 0 \text{ and } xy \neq 1. \]

**Solution:** True. For each \( x \), if \( x \neq 0 \) take \( y = 2/x \), so \( xy = 2 \neq 1 \) and \( y \neq 0 \). If \( x = 0 \) then take \( y = 1 \), so \( xy \neq 1 \) and \( y \neq 0 \).

(c) (5 marks) Let \( Q \) be the statement: \( \exists n \in \mathbb{N}, \text{ such that } \forall m \in \mathbb{N} \text{ we have } n \neq m + 5. \)

(i) Write \( \sim Q \).

(ii) Determine whether \( Q \) is true or false — Put True or False in the box, and **Justify your answers**.

**Solution:** \( \sim Q \) is: \( \forall n \in \mathbb{N}, \exists m \in \mathbb{N} \text{ such that } n = m + 5. \)

\( Q \) is true. Take \( n = 1 \). For every \( m \in \mathbb{N} \) we have \( m + 5 \geq 6 \) so \( m + 5 \neq 1 \).
2. Prove that the statements \( P \Rightarrow (Q \Rightarrow R) \) and \( (P \land Q) \Rightarrow R \) are logically equivalent, by the truth tables.

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Define \( f(x) = xe^{3x} \), and recall that \( f^{(n)}(x) \) denotes the \( n \)th derivative of \( f(x) \). Show that \( f^{(n)}(x) = (3^n x + n3^{n-1})e^{3x} \) for all \( n \in \mathbb{N} \).

**Solution:** We proceed by induction. For the base case, when \( n = 1 \), note that

\[
f^{(1)}(x) = f'(x) = \frac{d}{dx}(xe^{3x}) = x \frac{d}{dx}(e^{3x}) + e^{3x} \frac{d}{dx}(x) = x \cdot 3e^{3x} + e^{3x} = (3^1 x + 1 \cdot 3^0)e^{3x}
\]

as required. For the inductive step, let \( n \in \mathbb{N} \) and suppose that \( f^{(n)}(x) = (3^n x + n3^{n-1})e^{3x} \). Then

\[
f^{(n+1)}(x) = \frac{d}{dx}(f^{(n)}(x)) = \frac{d}{dx}((3^n x + n3^{n-1})e^{3x})
\]

\[
= (3^n x + n3^{n-1}) \frac{d}{dx}(e^{3x}) + e^{3x} \frac{d}{dx}(3^n x + n3^{n-1})
\]

\[
= (3^n x + n3^{n-1})(3e^{3x}) + e^{3x}(3^n) = (3^{n+1} x + (n + 1)3^n)e^{3x}
\]

as required.
4. Let $a, b \in \mathbb{Z}$. Prove that $3 \mid (a^2 + b^2)$ if and only if $3 \mid a$ and $3 \mid b$.

**Solution:** This is a biconditional statement. Thus, we need to prove both sides of the implications.

**Proof of (3 $\mid a$ and 3 $\mid b$ implies 3 $\mid (a^2 + b^2)$):** Assume that 3 $\mid a$ and 3 $\mid b$. Then we know $a = 3k$ and $b = 3l$ for some $k, l \in \mathbb{Z}$. Hence, $a^2 + b^2 = 9(k^2 + l^2)$. Therefore $3 \mid a^2 + b^2$.

**Proof of (3 $\mid (a^2 + b^2)$ implies 3 $\mid a$ and 3 $\mid b$):** We are going to use proof by contrapositive. Assume that 3 $\nmid a$ or 3 $\nmid b$. Then we have 2 cases. WLOG we can assume 3 $\nmid a$. Then we see that we have 2 cases $a \equiv 1 \pmod{3}$ or $a \equiv 2 \pmod{3}$.

**Case 1:** $a \equiv 1 \pmod{3}$: Then we see that $a^2 \equiv 1 \pmod{3}$.

**Case 2:** $a \equiv 2 \pmod{3}$: Then we see that $a^2 \equiv 4 \equiv 1 \pmod{3}$. Thus, we see that if 3 $\nmid a$, then $a^2 \equiv 1 \pmod{3}$. Thus, if 3 $\mid b$, then we get $b^2 \equiv 0 \pmod{3}$ and hence $a^2 + b^2 \equiv 1 \pmod{3}$. Moreover, if 3 $\nmid b$, then we get $b^2 \equiv 1 \pmod{3}$ and hence $a^2 + b^2 \equiv 2 \pmod{3}$. In both cases, we see that 3 $\nmid (a^2 + b^2)$. 
5. For \( a \in \mathbb{R} \), we define the set \( S_a = \{ x \in \mathbb{R} : x^2 \leq a^2 - 25 \} \). Show that if \( S_a = \emptyset \) then \( a \in (-5, 5) \).

**Solution:** We prove the contrapositive: For \( a \in \mathbb{R} \), if \( a \notin (-5, 5) \) then \( S_a \neq \emptyset \).

Let \( a \in \mathbb{R} \) be arbitrary, and assume that \( a \notin (-5, 5) \), which can be written equivalently as

\[
\sim(a \in (-5, 5)) \\
\sim(-5 < a < 5) \\
\sim((-5 < a) \land (a < 5)) \\
(-5 \geq a) \lor (a \geq 5).
\]

**Case 1:** Suppose \(-5 \geq a \). Squaring both sides (which reverses the inequality since the numbers are negative) yields \( 25 \leq a^2 \), and so \( 0 \leq a^2 - 25 \). If we set \( x = 0 \), then \( x \in \mathbb{R} \) and \( x^2 = 0 \leq a^2 - 25 \). In particular, \( x \in S_a \) by definition, and so \( S_a \neq \emptyset \).

**Case 2:** Suppose \( a \geq 5 \). Squaring both sides yields \( a^2 \geq 25 \), and so \( 0 \leq a^2 - 25 \); just as in Case 1 this gives \( 0 \in S_a \), and so \( S_a \neq \emptyset \).

In either case we have shown that \( S_a \neq \emptyset \), as required.

[Note: we could choose other values of \( x \), such as \( x = \sqrt{a^2 - 25} \), to show that \( S_a \neq \emptyset \). See solution below.]

**Solution:**

We prove the contrapositive. Assume \( a \notin (-5, 5) \). Then \( a^2 \geq 25 \), so \( a^2 - 25 \geq 0 \). For \( x = \sqrt{a^2 - 25} \in \mathbb{R} \), we have \( x^2 = (\sqrt{a^2 - 25})^2 \leq a^2 - 25 \). So \( x \in S_a \) and \( S_a \neq \emptyset \).
This page has been left blank for your workings and solutions.