The Simplex Method, including URS variables

Entering Variable:

1. Choose a nonbasic URS variable with a nonzero entry in the objective row. If the entry is negative, it enters increasing; if positive, it enters decreasing.

2. If there are none of those, choose an ordinary (≥ 0, but not artificial) variable with a negative entry in the objective. It enters increasing.

3. Artificial variables are never chosen as entering variable (except of course for an initial pivot that puts a_0 in the basis).

4. If there's a choice in (1) or in (2), we usually (if not using Bland's Rule) choose the entry that's largest in absolute value, and break ties by choosing the one farther left in the tableau.

Leaving Variable: Suppose x_E is the entering variable.

1. If x_E is increasing, calculate ratios (constant term)/(x_E entry) for non-URS basic variables with positive entries in the x_E column.

2. If x_E is decreasing, calculate ratios (constant term)/($-x_E$ entry) for non-URS basic variables with negative entries in the x_E column.

3. In Phase II, if there are basic artificial variables (which must have constant term 0), include the ratio 0 for these variables if the x_E entry is nonzero (either positive or negative).

4. The leaving variable is one with the minimum ratio. In case of a tie, artificial variables have first priority. Otherwise, if not using Bland's Rule, break ties by taking the candidate that is higher up in the tableau.

Worked example:

maximize $4x_1 - x_2 - x_3 + x_4$
subject to $3x_1 - x_2 - x_3 \leq -3$
$x_1 + 4x_4 \leq -2$
$-3x_1 + 2x_2 + x_3 - 2x_4 \leq 6$
$x_1 - 2x_2 = -2$
$x_1, x_2 \geq 0, x_3, x_4\ URS$

We change the last equation to $-x_1 + 2x_2 + a_4 = 2$, and subtract the artificial variable a_0 from the first two constraints. The temporary objective is $w = -a_4 - a_0$. The initial tableau is

w	z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	0	0	0	0	0	0	0	1	1	0	=	w
0	1	−4	1	1	−1	0	0	0	0	0	0	=	z
0	0	3	−1	−1	0	1	0	0	0	−1	−3	=	s_1
0	0	1	0	0	4	0	1	0	0	−1	−2	=	s_2
0	0	−3	2	1	−2	0	0	1	0	0	6	=	s_3
0	0	−1	2	0	0	0	0	0	1	0	2	=	a_4

Subtract the a_4 row from the w row to fix the a_4 column. The new w row is $[1, 0, 1, -2, 0, 0, 0, 0, 0, 0, 1, -2]$

Initial pivot to make a feasible bfs for the relaxed problem: a_0 enters, s_1 leaves.

w	z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	4	−3	−1	0	1	0	0	0	0	−5	=	w
0	1	−4	1	1	−1	0	0	0	0	0	0	=	z
0	0	−3	1	1	0	−1	0	0	0	1	3	=	a_0
0	0	−2	1	1	4	−1	1	0	0	0	1	=	s_2
0	0	−3	2	1	−2	0	0	1	0	0	6	=	s_3
0	0	−1	2	0	0	0	0	0	1	0	2	=	a_4

The URS variable x_3 has a negative entry in the w row, so it enters increasing. Ratios are 3/1 for s_1, 1/1 for s_2, 6/1 for s_3, so s_2 leaves.

w	z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	2	−2	0	4	0	1	0	0	0	−4	=	w
0	1	−2	0	0	−5	1	−1	0	0	0	−1	=	z
0	0	−1	0	0	−4	0	−1	0	0	1	2	=	a_0
0	0	−2	1	1	4	−1	1	0	0	0	1	=	x_3^U
0	0	−1	1	0	−6	1	−1	1	0	0	5	=	s_3
0	0	−1	2	0	0	0	0	0	1	0	2	=	a_4

The URS variable x_4 has a positive entry in the w row, so it enters decreasing. Ratios are 2/4 for a_0, 5/6 for s_2, so a_0 leaves.

w	z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	1	−2	0	0	0	0	0	0	1	−2	=	w
0	1	−3/4	0	0	0	1	1/4	0	0	−5/4	−7/2	=	z
0	0	1/4	0	0	1	0	1/4	0	0	−1/4	−1/2	=	x_4^U
0	0	−3	1	1	0	−1	0	0	0	1	3	=	x_3^U
0	0	1/2	1	0	0	1	1/2	1	0	−3/2	2	=	s_3
0	0	−1	2	0	0	0	0	0	1	0	2	=	a_4

Don't be alarmed by the negative value for x_4, because that's a URS variable. Now x_2 enters and a_4 leaves.

w	z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	0	0	0	0	0	0	0	1	1	0	=	w
0	1	$-3/4$	0	0	0	1	$1/4$	0	0	$-5/4$	$-7/2$	=	z
0	0	$1/4$	0	0	1	0	$1/4$	0	0	$-1/4$	$-1/2$	=	x_4^U
0	0	$-5/2$	0	1	0	-1	0	0	$-1/2$	1	2	=	x_3^U
0	0	1	0	0	0	1	$1/2$	1	$-1/2$	$-3/2$	1	=	s_3
0	0	$-1/2$	1	0	0	0	0	0	$1/2$	0	1	=	x_2

We've successfully concluded Phase I with $w = 0$, so we delete the w row and column and continue with the objective z. x_1 enters and s_3 leaves.

z	x_1	x_2	x_3^U	x_4^U	s_1	s_2	s_3	a_4	a_0	rhs		
1	0	0	0	0	$7/4$	$5/8$	$3/4$	$-3/8$	$-19/8$	$-11/4$	=	z
0	0	0	0	1	$-1/4$	$1/8$	$-1/4$	$1/8$	$1/8$	$-3/4$	=	x_4^U
0	0	0	1	0	$3/2$	$5/4$	$5/2$	$-7/4$	$-11/4$	$9/2$	=	x_3^U
0	1	0	0	0	1	$1/2$	1	$-1/2$	$-3/2$	1	=	x_1
0	0	1	0	0	$1/2$	$1/4$	$1/2$	$1/4$	$-3/4$	$3/2$	=	x_2

This tableau is optimal.