Sensitivity Analysis using the Dual Simplex Method

I will use as an example the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 2x_2 + x_3 - 3x_4 \\ \text{subject to} & 3x_1 + x_2 - x_4 \le 1 \\ & x_1 + x_2 + x_3 + x_4 \le 2 \\ & -3x_1 + 2x_3 + 5x_4 \le 6 \\ \text{all variables} & \ge 0 \end{array}$$

The optimal tableau is as follows:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	3	1	1	0	3	=	z
0	−2	0	1	2	−1	1	0	1	=	x_3
0	3	1	0	−1	1	0	0	1	=	x_2
0	1	0	0	1	2	−2	1	4	=	s_3

Thus we have

$$B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \\ s_3 \end{matrix}.$$

Each of the examples below will start from this basis (i.e. the changes made in one section will not affect later sections).

(1) Change in b

Suppose we change $\mathbf{b}$ to $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$. The new $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \\ s_3 \end{matrix}$. The new value of z in the basic solution is $\mathbf{y}^T\mathbf{b} = 5$, but the basic solution is not feasible. We need a dual simplex pivot: x_3 leaves, and (in a tie for minimum ratio) x_1 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	5	0	2	0	4	=	z
0	1	0	−1/2	−1	1/2	−1/2	0	1/2	=	x_1
0	0	1	3/2	2	−1/2	3/2	0	3/2	=	x_2
0	0	0	1/2	2	3/2	−3/2	1	11/2	=	s_3

This is now feasible and thus optimal.

(2) Parametric programming

We will see how the optimal solution depends on b_2. With $\mathbf{b} = \begin{pmatrix} 1 \\ p \\ 6 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} -1+p \\ 1 \\ 8-2p \end{pmatrix}$, so our basis x_3, x_2, s_3 is optimal for $1 \le p \le 4$. The z value is $1 + p$, and the tableau is

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	3	1	1	0	$1+p$	$=$	z
0	-2	0	1	2	-1	1	0	$-1+p$	$=$	x_3
0	3	1	0	-1	1	0	0	1	$=$	x_2
0	1	0	0	1	2	-2	1	$8-2p$	$=$	s_3

If $p < 1$, the tableau is not feasible: x_3 leaves and x_1 enters. The resulting tableau is

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	5	0	2	0	$2p$	$=$	z
0	1	0	$-1/2$	-1	$1/2$	$-1/2$	0	$(1-p)/2$	$=$	x_1
0	0	1	$3/2$	2	$-1/2$	$3/2$	0	$(-1+3p)/2$	$=$	x_2
0	0	0	$1/2$	2	$3/2$	$-3/2$	1	$(15-3p)/2$	$=$	s_3

This is feasible (and thus optimal) if $1/3 \leq p \leq 1$. If $p < 1/3$, x_2 leaves and s_1 enters:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	5	0	2	0	$2p$	$=$	z
0	1	1	1	1	0	1	0	p	$=$	x_1
0	0	-2	-3	-4	1	-3	0	$1-3p$	$=$	s_1
0	0	3	5	8	0	3	1	$6+3p$	$=$	s_3

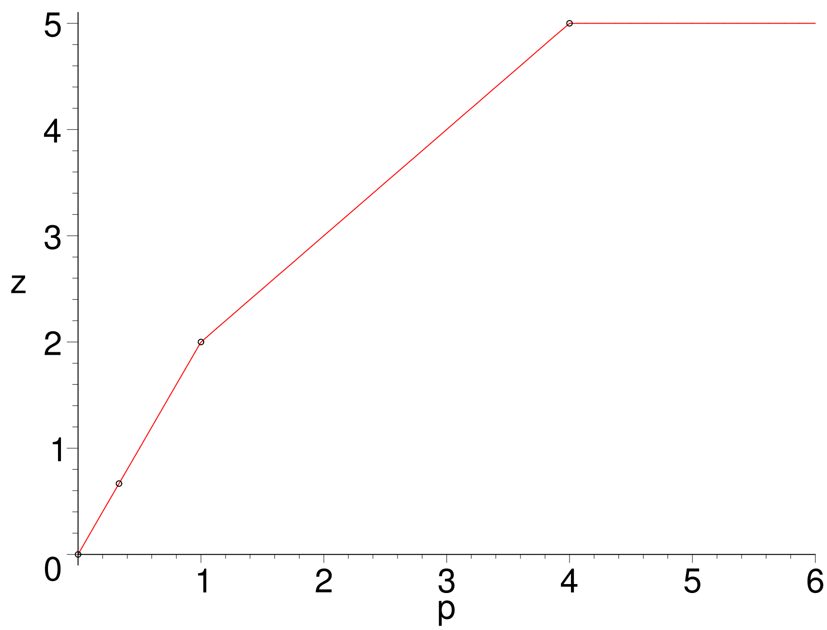
This is feasible, and thus optimal, if $0 \leq p \leq 1/3$. If $p < 0$, x_1 would have to leave but no variable could enter, and the problem would be infeasible.

Now on the other side, returning to the tableau that was optimal for $1 \leq p \leq 4$, if $p > 4$ we must let s_3 leave and s_2 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	$5/2$	0	0	$7/2$	2	0	$1/2$	5	$=$	z
0	$-3/2$	0	1	$5/2$	0	0	$1/2$	3	$=$	x_3
0	3	1	0	-1	1	0	0	1	$=$	x_2
0	$-1/2$	0	0	$-1/2$	-1	1	$-1/2$	$-4+p$	$=$	s_2

This is optimal if $p \geq 4$.

Plotting the optimal z as a function of p, we have



(3) Deleting a variable

Deleting a variable means requiring it to be 0. In effect, it becomes an artificial variable. If the variable is already nonbasic, nothing needs to be done. If it is basic, however, it must leave the basis. This can be done with a "sign-reversed Dual Simplex pivot". It's "sign-reversed" because the variable starts off with a positive value, rather than a negative value which is usual in the Dual Simplex method. We can make it look like an ordinary Dual Simplex pivot if we multiply the departing variable's row (except for the 1 for that basic variable) by -1 After the first pivot, we can remove the deleted variable from the problem. More Dual Simplex pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we want to delete the variable x_2, which is basic and has the value 1 in the optimal solution of the original problem. Here's the tableau with the x_2 row multiplied by -1:

z	x_1	$-x_2$	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	3	1	1	0	3	=	z
0	-2	0	1	2	-1	1	0	1	=	x_3
0	-3	1	0	1	-1	0	0	-1	=	$-x_2$
0	1	0	0	1	2	-2	1	4	=	s_3

$-x_2$ leaves, and x_1 enters.

z	x_1	$-x_2$	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	2/3	0	11/3	1/3	1	0	7/3	=	z
0	0	−2/3	1	4/3	−1/3	1	0	5/3	=	x_3
0	1	−1/3	0	−1/3	1/3	0	0	1/3	=	x_1
0	0	1/3	0	4/3	5/3	−2	1	11/3	=	s_3

This is feasible, so it is the new optimal solution.

(4) Deleting a constraint

Deleting a constraint means that we no longer care about the value of its slack variable. In effect, that slack variable has become sign-free. If the slack variable was already basic, no pivoting is necessary, but if it is nonbasic (with a nonzero entry in the z row) we will want it to enter the basis: increasing if the entry was negative (which would only happen if the constraint was an equality), or decreasing if the entry was positive. Recall that if a variable enters decreasing, we calculate ratios in the rows where the entry of the entering variable is negative. After the first pivot, we can delete the equation for the deleted constraint's slack variable from the tableau. More pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we delete the first constraint. Then s_1, which is nonbasic and has entry 1 in the z row, enters decreasing. The only ratio to be calculated is for x_3, so x_3 leaves the basis. The next tableau is

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	5	0	2	0	4	=	z
0	2	0	−1	−2	1	−1	0	−1	=	s_1
0	1	1	1	1	0	1	0	2	=	x_2
0	−3	0	2	5	0	0	1	6	=	s_3

We can delete the s_1 equation. The solution is now optimal.

(5) Adding a constraint

The new constraint has a new slack variable, and a new row of the tableau in which this slack variable is basic. That row comes from the equation for the new slack variable, but we need to substitute in the expressions for the basic variables so that it's expressed in terms of the nonbasic variables.

If the value of the new slack variable is negative (or if the new constraint is an equality and the value is nonzero), a Dual Simplex pivot will be needed.

In our example, we add the constraint $x_1 + 2x_2 + 2x_3 \leq 3$, or $x_1 + 2x_2 + 2x_3 + s_4 = 3$. Sustituting in the values of the basic variables x_3, x_2 and s_3, we get the new row of the tableau:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	rhs		
1	2	0	0	3	1	1	0	0	3	=	z
0	−2	0	1	2	−1	1	0	0	1	=	x_3
0	3	1	0	−1	1	0	0	0	1	=	x_2
0	1	0	0	1	2	−2	1	0	4	=	s_3
0	−1	0	0	−2	0	−2	0	1	−1	=	s_4

We need a dual simplex pivot, with s_4 leaving, and s_2 entering.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	rhs		
1	3/2	0	0	2	1	0	0	1/2	5/2	=	z
0	−5/2	0	1	1	−1	0	0	1/2	1/2	=	x_3
0	3	1	0	−1	1	0	0	0	1	=	x_2
0	2	0	0	3	2	0	1	−1	5	=	s_3
0	1/2	0	0	1	0	1	0	−1/2	1/2	=	s_2

This is optimal.

(6) Changing entries in A

There are two easy cases here, and one difficult case, depending on whether the constraints and variables where the entries are changed are basic or nonbasic.

(a) Suppose the changes are only in the coefficients of one or more nonbasic variables. Then it is as if the old versions of these variables were removed and new versions added. If the entries for the new versions in the z row are negative, pivots will be necessary.

In our example, suppose we change the coefficient of x_4 in the first constraint from -1 to -2. Think of this as introducing a new x_4. We calculate $\eta_4 = (1,1,0)\begin{pmatrix}-2\\1\\5\end{pmatrix} - (-3) = 2 \geq 0$, so the optimal solution is unchanged.

(b) Suppose the changes are only in a constraint whose slack variable is basic. Just as in "Adding a constraint", we express the new version of this slack variable in terms of the nonbasic variables. If the value of this slack variable is negative, one or more Dual Simplex pivots will be needed.

In our example, suppose we change the coefficient of x_2 in the third constraint from 0 to 2. The slack variable s_3 for the original version of the constraint was 4 in the basic solution, and x_2 was 1; the new slack variable $s_3' = s_3 - 2x_2$ will be $4 - 2 \times 1 = 2$.

This is still feasible, so no pivoting is necessary.

(c) Suppose the changes are for a basic variable and involve one or more constraints with nonbasic slack variables. Then the situation is more complicated. It is even possible that the B matrix may not be invertible (so that the current basis can no longer be used as a basis). Or if the basis can still be used, the solution may or may not be feasible for either the primal or the dual. One strategy that often works well is first to add a new version of the variable, and then delete the old version.

In our example, suppose we change the coefficient of x_2 in the second constraint from 1 to 2. We first add the new version of x_2, call it x_2', with coefficients 1, 2, 0 in the three constraints and 2 in the objective. Its η value is $(1,1,0)\begin{pmatrix}1\\2\\0\end{pmatrix} - 2 = 1 \geq 0$, so x_2' doesn't enter the basis. However, we need to put it into the tableau. We have $\mathbf{d} = B^{-1}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$, so the tableau is

z	x_1	x_2	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	1	0	3	1	1	0	3	=	z
0	−2	0	1	1	2	−1	1	0	1	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x_2
0	1	0	−2	0	1	2	−2	1	4	=	s_3

Now we delete the old x_2 in a "sign-reversed Dual Simplex" pivot. After changing signs in the x_2 equation, we make $-x_2$ leave. The ratios are $2/3$ for x_1, $1/1$ for x_2', $1/1$ for s_1, so x_1 enters. The new tableau is

z	x_1	$-x_2$	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	2/3	1/3	0	11/3	1/3	1	0	7/3	=	z
0	0	−2/3	5/3	1	4/3	−1/3	1	0	5/3	=	x_3
0	1	−1/3	1/3	0	−1/3	1/3	0	0	1/3	=	x_1
0	0	1/3	−7/3	0	4/3	5/3	−2	1	11/3	=	s_3

We delete the old x_2, and we have an optimal solution.

In some cases it would be useful to know if making the change will increase or decrease the objective value. Again, consider first adding the new version of the variable, and then deleting the old version. Note that primal simplex pivots increase the objective (or in the case of degeneracy may keep it the same, but never decrease it), while dual simplex pivots usually decrease the objective and never increase it.

- If the new version of the variable will not enter the basis when it's added, because its $\boldsymbol{\eta}$ value is not negative, then there is no primal simplex pivot but there will be at least one dual simplex pivot. So the change can't increase the objective, and will probably decrease it. This is what occurred in the example above, where the change decreased the objective from 3 to 7/3.
- If the new version of the variable does enter the basis because its $\boldsymbol{\eta}$ value is negative, the primal simplex pivots (if not degenerate) will increase the objective. If in this process the old version of the variable will leave the basis, there will be no need for dual simplex pivots to remove the old version, and so the objective can't decrease.

For example, suppose at the same time as we change the coefficient of x_2 in the second constraint from 1 to 2 we also increase c_2 (for the new version of the variable) from 2 to 4. Then η_2' decreases from 1 to -1, so the tableau is

z	x_1	x_2	x'_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	−1	0	3	1	1	0	3	=	z
0	−2	0	1	1	2	−1	1	0	1	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x_2
0	1	0	−2	0	1	2	−2	1	4	=	s_3

This time x'_2 will enter the basis. Since this won't be a degenerate pivot, the objective will increase. There is a tie for minimum ratio between x_3 and x_2; we may as well choose x_2 to leave (since it will be leaving anyway). At this point we know that no dual simplex pivots will be needed to make x_2 leave, so the change will increase the objective. In fact the next tableau is

z	x_1	x_2	x'_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	5	1	0	0	2	2	1	0	4	=	z
0	−5	−1	0	1	3	−2	1	0	0	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x'_2
0	7	2	0	0	−1	4	−2	1	6	=	s_3

which is optimal. We can then delete the column for the old x_2. The objective has increased from 3 to 4.