**Math 340 sec. 202: Solutions to Sample Midterm**

**1(a).** The initial basic solution is not feasible, so we need a Phase I. Adding an artificial variable $a_0$ and temporary objective tableau $w = -a_0$, the initial tableau is

| | $w$ | $z$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $a_0$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $z$ | 0 | 1 | $-1$ | $-3$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $s_1$ | 0 | 0 | 0 | 2 | $-2$ | 1 | 0 | 0 | $-1$ | $-2$ |
| $s_2$ | 0 | 0 | $-1$ | 0 | $-2$ | 0 | 1 | 0 | $-1$ | $-1$ |
| $s_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 2 |

Now $a_0$ enters the basis and the most negative slack variable, $s_1$, leaves: the result is a bfs for the relaxed problem.

| | $w$ | $z$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $a_0$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | 1 | 0 | 0 | 2 | $-2$ | 1 | 0 | 0 | 0 | $-2$ |
| $z$ | 0 | 1 | $-1$ | $-3$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $a_0$ | 0 | 0 | 0 | $-2$ | 2 | $-1$ | 0 | 0 | 1 | 2 |
| $s_2$ | 0 | 0 | $-1$ | $-2$ | 0 | $-1$ | 1 | 0 | 0 | 1 |
| $s_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 2 |

$x_3$ enters and $a_0$ leaves.

| | $w$ | $z$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $a_0$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $z$ | 0 | 1 | $-1$ | $-2$ | 0 | $1/2$ | 0 | 0 | $-1/2$ | $-1$ |
| $x_3$ | 0 | 0 | 0 | $-1$ | 1 | $-1/2$ | 0 | 0 | $1/2$ | 1 |
| $s_2$ | 0 | 0 | $-1$ | $-2$ | 0 | $-1$ | 1 | 0 | 0 | 1 |
| $s_3$ | 0 | 0 | 1 | 2 | 0 | $1/2$ | 0 | 1 | $-1/2$ | 1 |

The temporary objective value is now 0 so the bfs is feasible. We delete the $w$ row and column and the $a_0$ column, making the $z$ row the objective again. $x_2$ enters and $s_3$ leaves the basis.

| | $z$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| $x_3$ | 0 | $1/2$ | 0 | 1 | $-1/4$ | 0 | $1/2$ | $3/2$ |
| $s_2$ | 0 | 0 | 0 | 0 | $-1/2$ | 1 | 1 | 2 |
| $x_2$ | 0 | $1/2$ | 1 | 0 | $1/4$ | 0 | $1/2$ | $1/2$ |

This is optimal. The solution is $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/2$. *Note:* since the entry in the $z$ row and $x_1$ column is 0, this is not the only optimal solution.

**(b).** This is slightly embarassing. When this question was originally used on a midterm, the method we were using was slightly different: there was no artificial variable $a_0$, and the "first" pivot was equivalent to our second pivot, resulting in a bfs with $z = -1$. So the answer then was that the optimal value of the objective is at least $-1$. Our first pivot does not result in a bfs for the original problem, so we can't conclude anything yet about the optimal solution.

**(c).** If you changed the right side of the first constraint from $-2$ to $-1$, or the right side of the second or third constraint to $-2$, the result of our first pivot would be degenerate, i.e. there would be a 0 in the final column in the $s_2$ or $s_3$ row. (Note in the case of the third constraint that making this $-2$ would cause you to put a $-1$ in the $a_0$ column for this constraint)

**2.(a).** With $\mathbf{b} = \begin{bmatrix} 9 \\ t+18 \\ 1 \end{bmatrix}$, we would have $\beta = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 3+t \\ 1 \end{bmatrix}$. This would be feasible, so this basis would give the optimal solution, if $t \geq -3$.

**(b).** With the new $\mathbf{c}^T = [1, 8, 1, 3]$, we would have $\mathbf{c}_{BV}^T = [1, 0, 8]$, $\pi^T = \mathbf{c}_{BV}^T \mathbf{B}^{-1} = [1, 0, 0]$, $\overline{c}_3 = \pi^T \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} - 1 = 0$ and $\overline{c}_4 = \pi^T \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} - 3 = -1$. It would not be optimal, and $x_4$ would enter the basis. The rest of the $x_4$ column of the tableau would have $\mathbf{B}^{-1} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -14 \\ 15 \\ 2 \end{bmatrix}$, while $\beta = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ (either calculated as $\mathbf{B}^{-1}\mathbf{b}$, or taken from (a) with $t = 0$). The ratios are $3/15 = 1/5$ for $s_2$ and $1/2$ for $x_2$. The minimum ratio is $1/5$, so $s_2$ would leave the basis.

**3.** We detect that in Phase II by finding a tableau where a variable $x_e$ should enter the basis (it is not artificial, and has a negative entry in the $z$ row or is URS and has a nonzero entry in the $z$ row) but there is no variable to leave the basis (there are no ratios to calculate: if the entry for $x_e$ in the $z$ row is negative, there are no positive entries in the $x_e$ column in rows labelled by non-URS variables; or if $x_e$ is URS and its entry in the $z$ row is positive, there are no negative entries in the $x_e$ column in rows labelled by non-URS variables). Since we are in Phase II, we have a basic feasible solution. A negative entry in the $z$ row means that increasing the value of this $x_e$, keeping the other nonbasic variables at 0, will increase the value of the objective $z$. The lack of positive entries in the column means that doing this will not decrease the value of any of the non-URS basic variables, and so will never cause the solution to become infeasible. Thus arbitrarily large increases in the $z$ value are possible. Similarly if $x_e$ is URS and its entry in the $z$ row is positive, decreasing the value of $x_e$ will increase $z$, and the lack of negative entries in the column means that doing this will not decrease the value of any non-URS basic variables.

(To reduce complications, I probably should have stated in this problem that you could assume there were no URS variables.)