Marks

[8] **1.** Consider the following tableau for a standard primal linear programming problem.

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	0	p	0	5	3	14	$=$	z
0	1	q	0	1	0	1	$=$	x_1
0	0	r	1	2	1	5	$=$	x_3

For each of the following questions (separately), give an example of values of p, q and r to make the statement true.

(a) The basic solution for this tableau is the unique optimal solution.

Solution: any values with $p > 0$ will do.

(b) The basic solution for this tableau is optimal but is not the only optimal solution, and there is another optimal solution with $x_2 = 2$.

Solution: If $p = 0$, we can increase x_2 while keeping the other nonbasic variables 0, and the objective z won't be affected. Then $x_1 = 1 - qx_2$ and $x_3 = 5 - rx_2$, so we need $q \le 1/2$ and $r \le 5/2$.

(c) The basic solution for this tableau is not optimal, and the basic solution for the next tableau (using the usual Simplex Method) will have $x_2 = 2$, $x_3 = 1$ and $z = 20$.

Solution: Increasing x_2 while keeping the other nonbasic variables 0, we need $x_1 = 1 - qx_2$ to hit 0 at $x_2 = 2$, i.e. $q = 1/2$; $x_3 = 5 - rx_2 = 1$ when $x_2 = 2$, so $r = 2$; $z = 14 - px_2 = 20$ when $x_2 = 2$, so $p = -3$.

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[8] **2.** Solve the following linear programming problem, using the Simplex Method.

$$\begin{aligned}
\text{maximize } z = & -3x_1 + 3x_2 + 4x_3 \\
\text{subject to } & x_1 + x_2 + x_3 \le 30 \\
& -2x_1 + x_2 + 2x_3 \ge 12 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Solution: We multiply the second constraint by -1 to put it into $\le$ form; we'll have -12 on the right side, so we'll need to introduce an artificial variable a_0 and have a Phase I.
The initial tableau is

w	z	x_1	x_2	x_3	s_1	s_2	a_0	rhs		
1	0	0	0	0	0	0	1	0	=	w
0	1	3	−3	−4	0	0	0	0	=	z
0	0	1	1	1	1	0	0	30	=	s_1
0	0	2	−1	−2	0	1	−1	−12	=	s_2

The first pivot has a_0 enter the basis and the variable with the most (in this case, only) negative value, s_2, leave.

w	z	x_1	x_2	x_3	s_1	s_2	a_0	rhs		
1	0	2	−1	−2	0	1	0	−12	=	w
0	1	3	−3	−4	0	0	0	0	=	z
0	0	1	1	1	1	0	0	30	=	s_1
0	0	−2	1	2	0	−1	1	12	=	a_0

Now x_3, with the most negative value in the objective row, enters. The ratios are $30/1$ for s_1 and $12/2$ for a_0, so a_0 leaves.

w	z	x_1	x_2	x_3	s_1	s_2	a_0	rhs		
1	0	0	0	0	0	0	1	0	=	w
0	1	−1	−1	0	0	−2	2	24	=	z
0	0	2	1/2	0	1	1/2	−1/2	24	=	s_1
0	0	−1	1/2	1	0	−1/2	1/2	6	=	x_3

Since $w = 0$, we have a successful conclusion of Phase I and a feasible basic solution of the original problem. We can now remove the w row and column and use z as the objective. We can also remove the column for the artificial variable a_0, since it won't be allowed to enter the basis. s_2 enters and s_1 leaves.

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	7	1	0	4	0	120	=	z
0	4	1	0	2	1	48	=	s_2
0	1	1	1	1	0	30	=	x_3

This is optimal: $x_1 = x_2 = 0$, $x_3 = 30$, $s_1 = 0$, $s_2 = 48$, $z = 120$.

Continued on page 4

[8] **3.** Perform **one** iteration of the Dual Simplex Method on the following linear programming problem. What does the result tell you about the optimal value, if any, of the objective?

$$\begin{aligned}
\text{maximize } z = & -3x_1 - 4x_2 - x_3 \\
\text{subject to } & \quad x_1 + x_2 \le 3 \\
& -x_1 - 2x_2 + x_3 \le -1 \\
& -2x_1 - x_3 \le -2 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Solution: The initial tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	3	4	1	0	0	0	0	=	z
0	1	1	0	1	0	0	3	=	s_1
0	−1	−2	1	0	1	0	−1	=	s_2
0	−2	0	−1	0	0	1	−2	=	s_3

s_3, with the most negative value, leaves the basis. The ratios are 3/2 for x_1 and 1/1 for x_3, so x_3 enters. The next tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	1	4	0	0	0	1	−2	=	z
0	1	1	0	1	0	0	3	=	s_1
0	−3	−2	0	0	1	1	−3	=	s_2
0	2	0	1	0	0	−1	2	=	x_3

This tells you that $z \le -2$ in the optimal solution (if any), because the objective value decreases or stays the same in a dual-simplex pivot. Actually $z < -2$: since the current tableau is non-degenerate, the next pivot will decrease z.

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[10] **4.** Consider the linear programming problem

$$\begin{array}{lrrrl} \text{maximize} & 2x_1 & +x_2 & +4x_3 & \\ \text{subject to} & x_1 & +x_2 & +x_3 & \leq 13 \\ & 2x_1 & +3x_2 & +2x_3 & \leq 22 \\ & x_1 & +2x_2 & +x_3 & = 11 \\ & x_1, x_2, x_3 \geq 0 & & & \end{array}$$

Your friend claims that the optimal solution has $x_1 = x_2 = 0$, $x_3 = 11$. Confirm this using Complementary Slackness, and find all optimal solutions of the dual problem. DO NOT USE THE SIMPLEX METHOD, WHETHER PRIMAL OR DUAL, IN THIS QUESTION.

Solution:
We first calculate the values of the slack variables using the alleged optimal solution: $s_1 = 2$, $s_2 = 0$, $a_3 = 0$. Complementary Slackness tells us that $\eta_3 = 0$ and $y_1 = 0$. Note also that y_3, the dual variable corresponding to a_3, is URS.
The equations of the dual, using $\eta_3 = y_1 = 0$, are

$$\begin{aligned} 2y_2 + y_3 - \eta_1 &= 2 \\ 3y_2 + 2y_3 - \eta_2 &= 1 \\ 2y_2 + y_3 \qquad &= 4 \end{aligned}$$

From the third equation, $y_3 = 4 - 2y_2$. Then the first and second equations say $\eta_1 = 2$ and $\eta_2 = 7 - y_2$. To have this be feasible, we only need $0 \leq y_2 \leq 7$. Thus we have confirmed that the friend's solution was optimal; the optimal solutions of the dual problem are $y_1 = 0$, $y_2 = y_2$, $y_3 = 4 - 2y_2$, $\eta_1 = 2$, $\eta_2 = 7 - y_2$, $\eta_3 = 0$, for $0 \leq y_2 \leq 7$.

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5. A linear programming problem in standard form has the optimal tableau

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	0	3	0	5	3	14	=	z
0	1	−1	0	1	0	1	=	x_1
0	0	2	1	2	1	5	=	x_3

Note that each of the following parts is independent of the others, i.e. when doing one part you do not consider changes made in the other parts.

[7] (a) What would be the effect on the basic variables and the objective of adding a small number ϵ (positive or negative) to the right side of the first constraint of the original problem? For what interval of values of ϵ would this be true? If ϵ was slightly outside this interval, what variables would enter and leave the basis?

Solution: $\beta_{\text{new}} = B^{-1}\left(\mathbf{b}_{\text{old}} + \begin{pmatrix}\epsilon\\0\end{pmatrix}\right) = \beta_{\text{old}} + \begin{pmatrix}\epsilon\\2\epsilon\end{pmatrix}$ and $z_{\text{new}} = \mathbf{y}^T\mathbf{b}_{\text{new}} = z_{\text{old}} + 5\epsilon$. Thus the new values (using the current basis) would be $x_1 = 1+\epsilon$, $x_3 = 5+2\epsilon$, $z = 14+5\epsilon$. This would be optimal as long as it is feasible: for $x_1 \ge 0$ we need $\epsilon \ge -1$, for $x_3 \ge 0$ we need $\epsilon \ge -5/2$. Thus the interval is $-1 \le \epsilon < \infty$. If ϵ was slightly less than -1, we would need a Dual Simplex pivot where x_1 leaves the basis and x_2 (with the only negative entry in the x_1 row) enters the basis.

[5] (b) Suppose a new variable x_4 is introduced, with (in the original problem) coefficient p in the first constraint and $-p$ in the second, and coefficient 3 in the objective. What must be true about p if an optimal solution of the new problem has $x_4 > 0$?

Solution: $\eta_4 = (5\,3)\begin{pmatrix}p\\-p\end{pmatrix} - 3 = 2p - 3$. For an optimal solution to have $x_4 > 0$, we must have $\eta_4 \le 0$, i.e. $p \le 3/2$.

[6] (c) Suppose we introduce a new constraint $x_1 + x_3 \le 6$. What new row and column would be this introduce into the tableau? What would happen to the optimal solution?

Solution: The new column would be for the slack variable s_3 (with entries of 0 except in the new row). We start with

$$\begin{matrix} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \\ (0 & 1 & 0 & 1 & 0 & 0 & 1 & 6 = s_3) \end{matrix}$$

and subtract the x_1 and x_3 rows to make 0's in the basic x_1 and x_3 columns, obtaining the new row

$$\begin{matrix} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \\ (0 & 0 & -1 & 0 & -3 & -1 & 1 & 0 = s_3) \end{matrix}.$$

Since the right side is not negative, this is still feasible: the optimal solution doesn't change.

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[13] **6.** Consider the linear programming problem

$$\begin{array}{lrrrrrl} \text{maximize} & x_1 & +2x_2 & +4x_3 & +3x_4 & +2x_5 & \\ \text{subject to} & x_1 & +x_2 & -2x_3 & +x_4 & +x_5 & \leq 1 \\ & x_1 & +2x_2 & +3x_3 & -3x_4 & +3x_5 & \leq 12 \\ & x_1 & & -x_3 & & +x_5 & \leq 3 \\ & \multicolumn{6}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0} \end{array}$$

In the process of solving it with the Revised Simplex Method, suppose we reach the basis x_1, s_2, x_3 with

$$B^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ 4 & 1 & -5 \\ -1 & 0 & 1 \end{pmatrix}$$

Perform the next pivot, obtaining the new basis, B^{-1} and $\boldsymbol{\beta}$.

$$\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{matrix} x_1 \\ s2 \\ x_3 \end{matrix} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$\mathbf{y}^T = (1,0,4)B^{-1} = (-5,0,6)$. $\boldsymbol{\eta}^T_{NBV} = y^T N - c^T_{NBV} = \begin{matrix} x_2 & x_4 & x_5 & s_1 & s_3 \\ (-7 & -8 & -1 & -5 & 6) \end{matrix}$. With the most negative entry, x_4 enters.

$\mathbf{d} = B^{-1} \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$. With only one positive entry, we must have s_2 leave.

$$\left[\begin{array}{r|rrr|r} -1 & -1 & 0 & 2 & 5 \\ 1 & 4 & 1 & -5 & 1 \\ -1 & -1 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{r|rrr|r} 0 & 3 & 1 & -3 & 6 \\ 1 & 4 & 1 & -5 & 1 \\ 0 & 3 & 1 & -4 & 3 \end{array}\right]$$

New basis: x_1, x_4, x_2, new $B^{-1} = \begin{pmatrix} 3 & 1 & -3 \\ 4 & 1 & -5 \\ 3 & 1 & -4 \end{pmatrix}$, new $\boldsymbol{\beta} = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$.

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7. A small firm produces four coffee blends (C_1, C_2, C_3 and C_4), each of which consists of a mixture of Brazilian, Colombian and Peruvian coffees. The table below gives the proportions of the different coffees making up each blend, the profit per kilogram of each blend sold, and the maximum amount of each coffee available for this week's production. The firm would like to choose amounts of each blend to produce so as to maximize its profits.

		Coffee			
		Brazilian	Colombian	Peruvian	Profit ($) per kg
Blend	C_1	20%	40%	40%	0.8
	C_2	40%	50%	10%	0.6
	C_3	30%	30%	40%	0.4
	C_4	70%	20%	10%	0.5
Availability(kg)		8000	6400	6000	

The following file is submitted to LINDO:

```
max 0.8 x1 + 0.6 x2 + 0.4 x3 + 0.5 x4
st
p1) 0.2 x1 + 0.4 x2 + 0.3 x3 + 0.7 x4 <= 8000
p2) 0.4 x1 + 0.5 x2 + 0.3 x3 + 0.2 x4 <= 6400
p3) 0.4 x1 + 0.1 x2 + 0.4 x3 + 0.1 x4 <= 6000
end
```

The results are as follows:

```
.LP OPTIMUM FOUND AT STEP      3
.
.        OBJECTIVE FUNCTION VALUE
.
.        1)      13600.00
.
.  VARIABLE        VALUE          REDUCED COST
.        X1     12000.000000         0.000000
.        X2         0.000000         0.425000
.        X3         0.000000         0.225000
.        X4      8000.000000         0.000000

.       ROW   SLACK OR SURPLUS     DUAL PRICES
.       P1)         0.000000         0.166667
.       P2)         0.000000         1.916667
.       P3)       400.000000         0.000000
.
. NO. ITERATIONS=       3

. RANGES IN WHICH THE BASIS IS UNCHANGED:
.
.                            OBJ COEFFICIENT RANGES
. VARIABLE         CURRENT         ALLOWABLE        ALLOWABLE
.                    COEF          INCREASE         DECREASE
.       X1         0.800000         0.200000         0.360000
.       X2         0.600000         0.425000         INFINITY
```

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```
.        X3         0.400000          0.225000          INFINITY
.        X4         0.500000          2.300000          0.100000
.
.                            RIGHTHAND SIDE RANGES
.       ROW          CURRENT         ALLOWABLE         ALLOWABLE
.                      RHS            INCREASE          DECREASE
.        P1      8000.000000      14400.000000       2400.000000
.        P2      6400.000000        369.230743       4114.285645
.        P3      6000.000000          INFINITY        400.000000
```

[4] (a) Briefly explain the practical significance to the firm of each entry in the `RIGHTHAND SIDE RANGES` for P2.

Solution: "Current RHS" shows the entries of **b**, which are the available amounts of each type of coffee. If this available amount for one of the types were to increase by up to the "Allowable increase" or decrease by up to the "Allowable decrease", the optimal basis would be unchanged. Thus the firm would still only produce blends C_1 and C_4 and would use all available Brazilian and Columbian coffees. Within this range, the increase in profit (in dollars) per kilogram of additional availability would be given by the "dual price" for that row.

Answer the following questions using the computer output - do NOT try to solve the linear programming problem yourself.

[5] (b) Suppose, in addition to the 6400 kg of Colombian coffee available from the usual supplier, there is an opportunity to buy some Colombian coffee from another supplier, at a somewhat higher price. Under what conditions should the firm buy from that source? What can you say about how much should be bought?

Solution: The dual price of constraint P2 is 1.916667, so the firm should be willing to buy up to the "Allowable increase" of 369.23 kg of additional Colombian coffee at up to \$1.916667 above the regular price. It is possible that they should buy more, depending on the price, but we can't determine how much more from this output.

[5] (c) The firm is considering a change in the relative proportions of Brazilian and Colombian coffees in blend C_3. All other data, including the profit per kilogram, would remain the same. Might this be a good idea? If so, which of the two should be increased, and by how much?

Solution: Suppose the new C_3 used fraction p Brazilian, $0.6 - p$ Colombian and 0.4 Peruvian coffees instead of the current 0.3, 0.3, 0.4. We can "price it out": the new η_3 would be $.166667p + 1.916667(.6 - p) - .4 = -1.75p + .75$. In order for this to be worthwhile, i.e. for it to be profitable to produce some of this blend, the new η_3 would have to be negative, i.e. $p > 0.42857$ (or $3/7$). Thus the fraction of Brazilian coffee would have to increase by more than 0.12857.

[5] (d) An across-the-board increase in the prices of the four blends is being considered. This would raise the profit per kilogram of all four blends by the same positive amount p. For what values of p can we be sure that the optimal solution will not change?

Solution: We can use the 100% rule here, since we're thinking of changing the c values for several variables including basic ones. The fractions of the allowable increases will be $p/.2$, $p/.425$, $p/.225$ and $p/2.3$, so we can be sure the optimal solution won't change if $p(1/.2+1/.425+1/.225+1/2.3) = 12.23217p \le 1$, i.e. $p \le .08175$ approximately.

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8. Consider the nonlinear programming problem

$$\begin{aligned} \text{maximize } & x_1 + 2x_2 \\ \text{subject to } & x_1 + x_2 \le 1 \\ & x_1^2 + x_2^2 \le 5 \end{aligned}$$

[7] (a) State the Karush-Kuhn-Tucker conditions for this problem.

Solution:

$$\begin{aligned} 1 - \lambda_1 - 2\lambda_2 x_1 &= 0 \\ 2 - \lambda_1 - 2\lambda_2 x_2 &= 0 \\ x_1 + x_2 &\le 1 \\ x_1^2 + x_2^2 &\le 5 \\ \lambda_1(1 - x_1 - x_2) &= 0 \\ \lambda_2(5 - x_1^2 - x_2^2) &= 0 \\ \lambda_1, \lambda_2 &\ge 0 \end{aligned}$$

[5] (b) Find a solution to the Karush-Kuhn-Tucker conditions. *Hint:* λ_1 and λ_2 are both nonzero.

Solution: If λ_1 and λ_2 are nonzero, we must have $x_1+x_2 = 1$ and $x_1^2+x_2^2 = 5$. So $x_1^2+(1-x_1)^2 = 5$, a quadratic equation whose roots are -1 and 2. Thus either $x_1 = 2$ and $x_2 = -1$ or $x_1 = -1$ and $x_2 = 2$.
If $x_1 = 2$ and $x_2 = -1$ the first two KKT equations say $1 - \lambda_1 - 4\lambda_2 = 0$ and $2 - \lambda_1 + 2\lambda_2 = 0$, and then $\lambda_2 = -1/6$ which is not allowed.
If $x_1 = -1$ and $x_2 = 2$ the first two KKT equations say $1 - \lambda_1 + 2\lambda_2 = 0$ and $2 - \lambda_1 - 4\lambda_2 = 0$. The solution is $\lambda_1 = 4/3$, $\lambda_2 = 1/6$, which is allowed.

[4] (c) How can you be sure that the solution found in (b) is the global maximum?

Solution: The objective is concave and the feasible region is convex, so any local maximum is a global maximum.

The End

Be sure that this examination has 12 pages including this cover

The University of British Columbia

Solutions to Examinations - April 2003

Mathematics 340

Linear Programming

Dr. Israel

Closed book examination — Time: 2 $\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Section** ______________________

Special Instructions:

Allowed aids: A non-graphing, non-programmable calculator, and one sheet (two-sided) of notes.
Write your name at the top of each page. Write your answers in the space provided. Show your work.

Rules governing examinations

1. Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.

Speaking or communicating with other candidates.

Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

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1		8
2		8
3		8
4		10
5		18
6		13
7		19
8		16
Total		100