Marks

1. *Consider the following linear programming problem (P).*

$$\begin{array}{lrrrl} \text{maximize} & 5x_1 & +x_2 & -x_3 & \\ \text{subject to} & x_1 & -x_2 & -2x_3 & \le -3 \\ & x_1 & & & \le 2 \\ & 3x_1 & +x_2 & -x_3 & = -2 \\ & x_1, x_2, x_3 \ge 0 & & & \end{array}$$

[16] (a) *Solve it using the Simplex Method, finding optimal solutions for both (P) and its dual (D).*

We need a Phase I. Since the right side of the equality constraint is negative, we multiply it by -1. The slack variable a_3 for that constraint is artificial, and in addition we have the artificial variable a_0 to take care of the right side of the first constraint being negative. The temporary objective is $w = -a_3 - a_0$. The initial tableau is

w	z	x_1	x_2	x_3	s_1	s_2	a_3	a_0	rhs		
1	0	0	0	0	0	0	1	1	0	=	w
0	1	−5	−1	1	0	0	0	0	0	=	z
0	0	1	−1	−2	1	0	0	−1	−3	=	s_1
0	0	1	0	0	0	1	0	0	2	=	s_2
0	0	−3	−1	1	0	0	1	0	2	=	a_3

We need to adjust the w row since a_3 is supposed to be basic: subtract the a_3 row from it, obtaining

w	z	x_1	x_2	x_3	s_1	s_2	a_3	a_0	rhs		
1	0	3	1	−1	0	0	0	1	−2	=	w

The initial pivot to make the basic solution feasible for the relaxed problem has a_0 entering and s_1 leaving.

w	z	x_1	x_2	x_3	s_1	s_2	a_3	a_0	rhs		
1	0	4	0	−3	1	0	0	0	−5	=	w
0	1	−5	−1	1	0	0	0	0	0	=	z
0	0	−1	1	2	−1	0	0	1	3	=	a_0
0	0	1	0	0	0	1	0	0	2	=	s_2
0	0	−3	−1	1	0	0	1	0	2	=	a_3

With the most negative entry in the w row, x_3 enters. The minimum ratio is for a_0, which leaves.

w	z	x_1	x_2	x_3	s_1	s_2	a_3	a_0	rhs		
1	0	5/2	3/2	0	−1/2	0	0	3/2	−1/2	=	w
0	1	−9/2	−3/2	0	1/2	0	0	−1/2	−3/2	=	z
0	0	−1/2	1/2	1	−1/2	0	0	1/2	3/2	=	x_3
0	0	1	0	0	0	1	0	0	2	=	s_2
0	0	−5/2	−3/2	0	1/2	0	1	−1/2	1/2	=	a_3

Now s_1 enters, and a_3 leaves.

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w	z	x_1	x_2	x_3	s_1	s_2	a_3	a_0	rhs		
1	0	0	0	0	0	0	1	1	0	=	w
0	1	−2	0	0	0	0	−1	0	−2	=	z
0	0	−3	−1	1	0	0	1	0	2	=	x_3
0	0	1	0	0	0	1	0	0	2	=	s_2
0	0	−5	−3	0	1	0	2	−1	1	=	s_1

We have successfully concluded Phase I, so we can delete the w row and column, as well as the a_0 column, and continue Phase II with z as objective. x_1 enters and s_2 leaves.

z	x_1	x_2	x_3	s_1	s_2	a_3	rhs		
1	0	0	0	0	2	−1	2	=	z
0	0	−1	1	0	3	1	8	=	x_3
0	1	0	0	0	1	0	2	=	x_1
0	0	−3	0	1	5	2	11	=	s_1

This is optimal (since a_3 is artificial, its negative entry in the z row does not matter). The optimal solution of (P) is $x_1 = 2$, $x_2 = 0$, $x_3 = 8$, $s_1 = 11$, $s_2 = a_3 = 0$. The optimal solution of (D) is $y_1 = 0$, $y_2 = 2$, $y_3 = -1$, $\eta_1 = \eta_2 = \eta_3 = 0$.

[4] (b) *Is there more than one optimal solution of (P)? Give reasons.*
Yes: since the entry in the z row for the nonbasic variable x_2 is 0, x_2 can be increased without affecting the objective.

[9] **2.** *Someone claims that $x_1 = 10$, $x_2 = 0$, $x_3 = 1$ is an optimal solution to the problem*

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} & 3x_1 + x_2 + 3x_3 \le 45 \\ & x_1 + x_2 \le 10 \\ & x_1 + 2x_2 + 5x_3 \le 15 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

Use complementary slackness to check whether the claim is correct.
With those values of x_1, x_2 and x_3 we have $s_1 = 45-30-3 > 0$, $s_2 = 10-10 = 0$, $s_3 = 15-10-5 = 0$. So the given solution is feasible for the primal problem, and complementary slackness says $y_1 = \eta_1 = \eta_3 = 0$ for the dual problem. The equations of the dual, with $y_1 = \eta_1 = \eta_3 = 0$, are

$$\begin{aligned} y_2 + y_3 &= 2 \\ y_2 + 2y_3 - \eta_2 &= 3 \\ 5y_3 &= 4 \end{aligned}$$

Thus $y_3 = 4/5$, $y_2 = 2 - 4/5 = 6/5$, and $\eta_2 = 6/5 + 8/5 - 3 = -1/5$. This is not feasible for the dual problem, so the claim is incorrect.

3. *Consider the following linear programming problem, and its optimal tableau :* (to fit our notation, I changed the "dictionary" of the original exam to a tableau)

maximize $4x_1 + x_2 + 5x_3 + 3x_4$

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$$\begin{array}{lrrrrl} \text{subject to} & x_1 & -x_2 & -x_3 & +3x_4 & \le 1 \\ & 5x_1 & +x_2 & +3x_3 & +8x_4 & \le 55 \\ & -x_1 & +2x_2 & +3x_3 & -5x_4 & \le 3 \\ & \text{all } x_i \ge 0 \end{array}$$

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	1	0	2	0	11	0	6	29	=	z
0	1	0	1	1	2	0	1	5	=	x_4
0	2	1	4	0	5	0	3	14	=	x_2
0	−5	0	−9	0	−21	1	−11	1	=	s_2

[9] (a) *If the right side of the third constraint is changed from 3 to p, for what values of the parameter p will the current basis be optimal? What is the optimal solution when p is the maximum value in this interval? For p slightly larger, what variables would enter and leave the basis?*

Note that B^{-1} is found in the tableau. The new $\boldsymbol{\beta} = \begin{pmatrix} 2 & 0 & 1 \\ 5 & 0 & 3 \\ -21 & 1 & -11 \end{pmatrix} \begin{pmatrix} 1 \\ 55 \\ p \end{pmatrix} = \begin{matrix} x_4 \\ x_2 \\ s_2 \end{matrix} \begin{pmatrix} 2+p \\ 5+3p \\ 34-11p \end{pmatrix}$.

The current basis is optimal as long as all these ≥ 0, i.e. $-5/3 \le p \le 34/11$. For $p = 34/11$, the optimal solution is $x_4 = 56/11$, $x_2 = 157/11$, $x_1 = x_3 = s_1 = s_2 = s_3 = 0$.

[6] (b) *If the first constraint is removed, what variables would enter and leave the basis on the first pivot?*

Removing the first constraint means making s_1 into a URS variable. Then s_1 will enter the basis decreasing. When a variable enters decreasing, we calculate ratios using the negative entries in its column. There is only one of these, for s_2, so s_2 would leave the basis.

[3] (c) *By how much would the coefficient of x_3 in the objective have to be increased to make the optimal solution have $x_3 > 0$?*

In the optimal tableau, x_3 is nonbasic with reduced cost $\eta_3 = 2$. Thus c_3 would have to be increased by at least 2 to make an optimal solution have $x_3 > 0$.

[6] **4.** *What does it mean for a linear programming problem to be unbounded? How would this fact be detected in the course of attempting to solve it using the Simplex Method?*

A linear programming problem is unbounded if there are feasible solutions with arbitrarily large positive objective values (for a maximize problem) or arbitrarily large negative objective values (for a minimize problem). This is detected in Phase II of the (primal) Simplex Method when there is an entering variable but no ratios to calculate and thus no leaving variable.

5. *Farmer Jones grows rutabagas and zucchini on a 45 hectare farm. He can sell up to 140 tonnes of rutabagas, at a profit of \$30 per tonne (excluding labour costs), and up to 120 tonnes of zucchini, at a profit of \$50 per tonne (excluding labour costs). Each hectare yields 5 tonnes of rutabagas or 4 tonnes of zucchini. Jones must hire 6 hours of labour to harvest each hectare of rutabagas, and 10 hours of labour to harvest each hectare of zucchini. Labour costs \$10 per hour, and at most 350 hours of labour is available. Being a very high-tech farmer, Jones uses a LINDO program, shown below together with its output, to decide how many hectares*

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to plant with each crop (x_1 for rutabagas, x_2 for zucchini) and how much labour to hire (x_3 hours).

```
max 150 x1 + 200 x2 - 10 x3
subject to
land) x1 + x2 <= 45
labour) 6 x1 + 10 x2 - x3 <= 0
maxlabour) x3 <= 350
maxruta) 5 x1 <= 140
maxzucc) 4 x2 <= 120
end
```

```
LP OPTIMUM FOUND AT STEP      4
OBJECTIVE FUNCTION VALUE
1)      4250.000

VARIABLE   VALUE      REDUCED COST
      X1   25.000000     0.000000
      X2   20.000000     0.000000
      X3  350.000000     0.000000

      ROW  SLACK OR SURPLUS  DUAL PRICES
    LAND)  0.000000            75.000000
  LABOUR)  0.000000            12.500000
MAXLABOU)  0.000000             2.500000
 MAXRUTA)  15.000000            0.000000
 MAXZUCC)  40.000000            0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:
   OBJ COEFFICIENT RANGES
VARIABLE     CURRENT  ALLOWABLE  ALLOWABLE
                COEF   INCREASE   DECREASE
      X1  150.000000  10.000000  30.000000
      X2  200.000000  50.000000  10.000000
      X3  -10.000000   INFINITY   2.500000

   RIGHTHAND SIDE RANGES
     ROW     CURRENT  ALLOWABLE  ALLOWABLE
                 RHS   INCREASE   DECREASE
    LAND   45.000000   1.200000   6.666667
  LABOUR    0.000000  40.000000  12.000000
MAXLABOU  350.000000  40.000000  12.000000
 MAXRUTA  140.000000   INFINITY  15.000000
 MAXZUCC  120.000000   INFINITY  40.000000
```

[6] (a) *What is the most that Jones should be willing to pay for an additional hour of labour? How much should he be willing to hire at that price?*

The "dual price" of the MAXLABOUR constraint is 2.5, so he should be willing to pay up to \$2.50 extra per hour to hire additional labour. Since he is now paying \$10 per hour for labour, that means he should be willing to pay up to \$12.50 per hour. You could also get this from the dual price of the LABOUR constraint. At that price, the basis remains optimal if the right side of the

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LABOUR or MAXLABOUR constraint is increased by up to 40, so he should be willing to hire at least 40 additional hours of labour at that price.

[4] (b) *If 5 hectares of land were removed from the farm to build a road, what would be the effect on the farm's profit?*

The dual price of the LAND constraint is 75. Since 5 is less than the allowable decrease for the right side of the LAND constraint, the basis remains optimal, and the profit would decrease by $5 \times 75 = \$375$.

[4] (c) *Explain the practical significance of the "Allowable decrease" for X2 in the "Obj coefficient ranges".*

The coefficient of X2 in the objective represents the contribution to the profit of one hectare planted in zucchini (at \$50 per tonne times 4 tonnes per hectare). If this were to decrease by up to the allowable decrease of \$10 per hectare, e.g. by a decrease in the selling price of zucchini or in the yield, the current solution would remain optimal.

[6] (d) *Jones is considering planting pumpkins. Demand for pumpkins is unlimited. Each hectare planted with pumpkins yields 4 tonnes of pumpkins, at a profit of \$30 per tonne (excluding labour costs), and requires 3 hours of labour. Should Jones plant any pumpkins?*

We must price out a new variable. The contribution of one hectare of pumpkins to the profit is $30 \times 4 = \$120$. The resources consumed are one hectare of land at \$75 and 3 hours of labour at \$12.50, for a total shadow price of \$112.50. Thus a hectare of pumpkins would more than pay for the resources it consumes. He should plant some pumpkins.

6. *Consider the following linear programming problem (P).*

$$\begin{array}{lrrrrr}
\text{maximize} & x_1 & +7x_2 & -3x_3 & & \\
\text{subject to} & x_1 & +2x_2 & & \le & 4 \\
 & & x_2 & -x_3 & \le & -2 \\
 & -2x_1 & -4x_2 & +x_3 & \le & -3 \\
 & \multicolumn{5}{l}{x_1, x_2, x_3 \ge 0}
\end{array}$$

In the process of solving this by the Revised Simplex Method, suppose we reach the basis x_1, s_3, x_3 with

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

[3] (a) *Check that this B^{-1} is correct.*

$$B^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[8] (b) *Perform the next pivot, obtaining the new basis, B^{-1} and $\boldsymbol{\beta}$.*

With $\mathbf{c}_{BV} = \overset{\begin{matrix} x_1 & s_3 & x_3 \end{matrix}}{\begin{pmatrix} 1 & 0 & -3 \end{pmatrix}}$, $\mathbf{y}^T = \mathbf{c}_{BV} B^(-1) = [1,3,0]$ and $\eta_2 = [1,3,0] \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} - 3 = -7$.

Thus x_2 enters. The x_2 column is

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$B^{-1}\begin{pmatrix}2\\1\\-4\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ while $\boldsymbol{\beta} = B^{-1}\begin{pmatrix}4\\-2\\-3\end{pmatrix} = \begin{matrix}x_1\\s_3\\x_3\end{matrix}\begin{pmatrix}4\\3\\2\end{pmatrix}$, so ratios are $\begin{matrix}2\\3\\---\end{matrix}$. With the least ratio, x_1 leaves. The update step is

$$\left[\begin{array}{r|rrr|r} 2 & 1 & 0 & 0 & 4\\ 1 & 2 & 1 & 1 & 3\\ -1 & 0 & -1 & 0 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{r|rrr|r} 1 & 1/2 & 0 & 0 & 2\\ 0 & 3/2 & 1 & 1 & 1\\ 0 & 1/2 & -1 & 0 & 4 \end{array}\right]$$

The new basis is x_2, s_3, x_3, $B^{-1} = \begin{pmatrix}1/2 & 0 & 0\\ 3/2 & 1 & 1\\ 1/2 & -1 & 0\end{pmatrix}$, and $\boldsymbol{\beta} = \begin{pmatrix}2\\1\\4\end{pmatrix}$.

[4] (c) *By changing the value of c_2, the coefficient of x_2 in the objective, we could obtain a problem for which the basis (x_1, s_3, x_3) is optimal. For what range of values of c_2 would this be true?*

We still have the same $\mathbf{y}$, and only η_2 will change: $\eta_2 = [1,3,0]\begin{pmatrix}-1\\-1\\3\end{pmatrix} - c_2 = -4 - c_2$. For this basis to be optimal we would need $c_2 \le -4$.

Note: the final question is on a topic we did not cover this term.

7. Suppose a two-person zero-sum game has the following payoff matrix for the row player, where t is a parameter:

$$\begin{array}{c} \\ (1)\\(2)\\(3)\end{array}\begin{array}{c}\begin{array}{ccc}(1) & (2) & (3)\end{array}\\ \begin{pmatrix}3 & -1 & 2\\ 1 & 0 & 4\\ 0 & t & 3\end{pmatrix}\end{array}$$

[6] (a) Write down a linear programming problem whose optimal solution gives an optimal strategy for the row player. DO NOT SOLVE.

[6] (b) Suppose that the column player's optimal strategy is $(1/2, 1/2, 0)$, and the row player has an optimal strategy that involves sometimes playing (1) (with probability greater than 0

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but less than 1). What can you say about the value of t? *Hint:* the row player still uses the same mixed strategy even if she knows the column player's mixed strategy.

The End

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The University of British Columbia

Sessional Examinations - April 2002

Mathematics 340 Section 201

Linear Programming

Dr. Israel

Closed book examination

Time: 2 $\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Section** ______________________

Special Instructions:

Allowed aids: One sheet (two-sided) of notes; non-graphing, non-programmable calculator.

Write your name at the top of each page. Write your answers in the space provided.

Show your work.

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1		20
2		9
3		18
4		6
5		20
6		15
7		12
Total		100