The Revised Simplex Method (Primal Phase II)

The standard-form problem is stated as: maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \ge 0$. However, we can allow equality constraints (corresponding to slack variables that are artificial) and sign-free variables: the modifications are that artificial variables are not allowed to enter the basis, sign-free variables are not allowed to leave the basis, and sign-free variables could enter decreasing as well as increasing. However, I'll assume (for the sake of simplicity) that all artificial variables are already nonbasic, and all sign-free variables are already basic. Then the only consideration we have to give to these is that artificial variables are ignored when we look for candidates to enter the basis, and sign-free variables are ignored when we look for candidates to leave the basis.

1. We start with a basis whose basic solution is feasible. That will continue to be true for all subsequent bases. We could begin with any such basis, but usually we start with the basis consisting of the slack variables, for which B is the identity matrix I. Otherwise we will have to calculate B^{-1}. B consists of the columns of A corresponding to the basic variables (in their order in the basis). The values of the basic variables in the current basic solution are given by $\boldsymbol{\beta} = B^{-1}\mathbf{b}$. Of course if $B = I$ this is just $\mathbf{b}$.
2. Calculate $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1}$ and $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - c_{NBV}^T$. Here $\mathbf{c}_{BV}$ is the vector consisting of the entries of $\mathbf{c}$ corresponding to the basic variables (0 for any slack variable), N and $\mathbf{c}_{NBV}$ are the columns of A and entries of $\mathbf{c}$ corresponding to nonbasic variables. No actual computation has to be done to get the entries of $\boldsymbol{\eta}_{NBV}$ for the nonbasic slack variables, as these will just be the corresponding entries of $\mathbf{y}$. The candidates for entering variable are the non-artificial variables corresponding (by complementary slackness) to negative entries in $\boldsymbol{\eta}_N$. If there are none, the current solution is optimal. If there are several, we usually take the most negative one (Largest Coefficient Rule).
3. Calculate $\mathbf{d} = B^{-1}A_e$, where A_e is the column of A corresponding to the entering variable. Calculate the ratios β_i/d_i where $d_i > 0$ (and the i'th basic variable is not sign-free). If there are no ratios to calculate, stop and conclude the problem is unbounded. Otherwise, the minimum ratio corresponds to the leaving variable.
4. Pivot, updating B^{-1} and $\boldsymbol{\beta}$ as follows. Form the matrix consisting of $\mathbf{d}$, B^{-1} and $\boldsymbol{\beta}$. Divide the row corresponding to the leaving variable by the entry of $\mathbf{d}$ in that row, so that this entry becomes 1. Add suitable multiples of this row to the other rows so the other entries in $\mathbf{d}$ become 0. Then the second to second-last columns of this matrix will form the new B^{-1}, and the last column will be the new $\boldsymbol{\beta}$. Return to step 2.

Worked example:

maximize $x_2 + x_3 + x_4 - 2x_5$
subject to $3x_1 + x_2 - x_5 \le 1$
$x_1 + x_2 + x_3 + x_4 \le 2$
$-3x_1 + 2x_3 + x_4 + 5x_5 \le 6$
all variables ≥ 0

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & s_3 \end{matrix} \\ \begin{pmatrix} 3 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 2 & 1 & 5 & 0 & 0 & 1 \end{pmatrix} \end{array}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \mathbf{c}^T = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & s_3 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & -2 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

1. The initial basis is s_1, s_2, s_3 so $B = I$ and $\boldsymbol{\beta} = \mathbf{b} = \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$. Thus the basic solution is $s_1 = 1$, $s_2 = 2$, $s_3 = 6$. Note that this is feasible, so we can start Phase II.

2. $\mathbf{c}_{BV}^T = \begin{pmatrix} \overset{s_1}{0} & \overset{s_2}{0} & \overset{s_3}{0} \end{pmatrix}$ so $\mathbf{y}^T = (0,0,0)$, $\boldsymbol{\eta}_{NBV} = 0 - \mathbf{c}_{NBV}^T = \begin{pmatrix} \overset{x_1}{0} & \overset{x_2}{-1} & \overset{x_3}{-1} & \overset{x_4}{-1} & \overset{x_5}{2} \end{pmatrix}$. There is a 3-way tie for most negative entry: I'll choose x_2 to enter.

3. $\mathbf{d} = B^{-1} \overset{x_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} = \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, ratios (using $\boldsymbol{\beta}$ from step 1) $\begin{matrix} 1/1 = 1 \\ 2/1 = 2 \\ --- \end{matrix}$. The minimum ratio 1 corresponds to s_1, so s_1 leaves.

4. Pivot and update: $\left[\begin{array}{c|ccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$ where we just had to subtract the first row from the second row. So now $B^{-1} = \begin{matrix} x_2 \\ s_2 \\ s_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{\beta} = \begin{matrix} x_2 \\ s_2 \\ s_3 \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$.

2. $\mathbf{y}^T = c_{BV}^T B^{-1} = \begin{pmatrix} \overset{x_2}{1} & \overset{s_2}{0} & \overset{s_3}{0} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1,0,0)$ and

$$\boldsymbol{\eta}_{NBV} = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = (1,0,0) \begin{pmatrix} \overset{x_1}{3} & \overset{x_3}{0} & \overset{x_4}{0} & \overset{x_5}{-1} & \overset{s_1}{1} \\ 1 & 1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 5 & 0 \end{pmatrix} - \begin{pmatrix} \overset{x_1}{0} & \overset{x_3}{1} & \overset{x_4}{1} & \overset{x_5}{-2} & \overset{s_1}{0} \end{pmatrix} =$$

$\begin{pmatrix} \overset{x_1}{3} & \overset{x_3}{-1} & \overset{x_4}{-1} & \overset{x_5}{1} & \overset{s_1}{1} \end{pmatrix}$. Again a tie: I'll let x_3 enter.

3. $\mathbf{d} = B^{-1} \overset{x_3}{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}} = \begin{matrix} x_2 \\ s_2 \\ s_3 \end{matrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, ratios $\begin{matrix} --- \\ 1/1 \\ 6/2 \end{matrix}$. The minimum ratio 1 corresponds to s_2, so s_2 leaves.

4. Pivot and update: $\left[\begin{array}{c|ccc|c} 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc|c} 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 1 & 4 \end{array}\right]$. So now $B^{-1} = \begin{matrix} x_2 \\ x_3 \\ s_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$ and $\boldsymbol{\beta} = \begin{matrix} x_2 \\ x_3 \\ s_3 \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$.

2. $\mathbf{y}^T = (1,1,0)B^{-1} = (0,1,0)$ and $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{pmatrix} \overset{x_1}{1} & \overset{x_4}{0} & \overset{x_5}{2} & \overset{s_1}{0} & \overset{s_2}{1} \end{pmatrix}$. No negative entries, so we have an optimal solution: $x_2 = 1$, $x_3 = 1$, $x_1 = x_4 = x_5 = s_1 = s_2 = 0$, $s_3 = 4$, $z = \mathbf{y}^T \mathbf{b} = 2$.