Newton's Method

Newton's Method in general is an iterative method for approximately solving systems of equations. In this case we use it to find critical points. Suppose we want to find a critical point of a function $f(x_1, ..., x_n)$. That is, we want to find a point $\mathbf{x}$ where the gradient $\nabla f(\mathbf{x}) = 0$. Let $H(\mathbf{x})$ be the Hessian matrix of f at $\mathbf{x}$. Then Newton's method is the iteration

$$\mathbf{x}_{k+1} = \mathbf{x}_k - H(\mathbf{x}_k)^{-1} \nabla f(\mathbf{x}_k)$$

Here $\mathbf{x}_k$ and $\nabla f(\mathbf{x}_k)$ are considered as column vectors. We start out at an initial guess $\mathbf{x}_0$, and use this formula to compute $\mathbf{x}_1$, $\mathbf{x}_2$, ... If we made a good initial guess, these will converge rapidly towards a solution. We stop when we think we are close enough to the solution (typically signalled by the fact that we have very little change from one iteration to the next).

Example:

$f(x_1, x_2) = x_1^2 + x_2 + x_1 x_2^2$.

$\nabla f(x_1, x_2) = (2x_1 + x_2^2, 1 + 2x_1 x_2)$.

$H(x_1, x_2) = \begin{pmatrix} 2 & 2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}$.

Suppose we start with $\mathbf{x}_0 = (1, 0)$: $\nabla f(\mathbf{x}_0) = (2, 1)$ and $H(\mathbf{x}_0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ so $H(\mathbf{x}_0)^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$$

For the next iteration, $\nabla f(\mathbf{x}_1) = (1/4, 1)$, $H(\mathbf{x}_1) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ so $H(\mathbf{x}_1)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$ and

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7/4 \end{pmatrix}$$

So far there's not much sign of convergence. However, if we persevere:

$$\begin{aligned}
\mathbf{x}_3 &= (0.3181818182, 0.693181818) \\
\mathbf{x}_4 &= (-1.664298114, 2.747545566) \\
\mathbf{x}_5 &= (-0.8309365361, 1.676201448) \\
\mathbf{x}_6 &= (-0.5508510643, 1.166731310) \\
\mathbf{x}_7 &= (-0.5011690966, 1.012915366)
\end{aligned}$$

Now maybe we're getting somewhere: this is not very far from $\mathbf{x}_6$. In fact, convergence now becomes very rapid:

$$
\begin{aligned}
\mathbf{x}_8 &= (-0.4999826445, 1.000065205) \\
\mathbf{x}_9 &= (-0.4999999984, 1.000000001) \\
\mathbf{x}_{10} &= (-0.5, 1.0)
\end{aligned}
$$

(to more than 10 decimal places accuracy). So it looks like we have found our critical point: $(-1/2, 1)$.