Math 340 Section 101: Solutions to Second Midterm

Nov. 15, 2006

Instructions: Non-programmable, non-graphing calculator and one sheet of notes (two-sided) allowed, no other aids. Remember to write your full name and student number on the front of your booklet.

1. [**16 pts**] *A factory produces knives, forks and spoons. The profits per item are \$2.00, \$3.00 and \$1.00 respectively. In deciding how many of each to produce each day, the manager solves the following linear programming problem:*

$$\begin{array}{ll} \text{maximize} & 2x_1+3x_2+x_3 \\ \text{subject to} & -x_1-x_2-x_3 \le -3000 \\ & \quad\;\; -x_2-x_3 \le -1080 \\ & 2x_1+3x_2+2x_3 \le 7000 \\ & x_1,x_2,x_3 \ge 0 \end{array}$$

where the 7000 in the third constraint is the available amount of a certain resource R_3. The manager obtains the following optimal tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	0	0	2	1	2	6920	=	z
0	1	0	0	-1	1	0	1920	=	x_1
0	0	1	0	2	0	1	1000	=	x_2
0	0	0	1	-2	-1	-1	80	=	x_3

(a) *If the factory could obtain more of resource R_3, what should they be willing to pay per unit for small amounts of it? How much would they be willing to buy at that price?*

The amount they should be willing to pay is the shadow price for the third constraint, obtained from the objective row of the tableau under s_3, namely \$2.00 per unit. This is in addition to any "regular" price for this resource that might be included in the problem formulation. The amount they should be willing to buy at that price is the maximum increase in b_3 for which the current basis gives the optimal solution. If $\mathbf{\Delta b} = [0,0,\epsilon]^T$, $\Delta\boldsymbol{\beta} = B^{-1}\Delta\mathbf{b}$, so the new $\boldsymbol{\beta}$ is $[1920, 1000+\epsilon, 80-\epsilon]$. For this to be feasible we need $\epsilon \le 80$. Thus they should be willing to buy up to 80 additional units at that price.

(b) *The market for forks has suddenly collapsed, so forks can't be produced any more. What would be the new optimal solution?*

We need to delete the basic variable x_2. This will require a "sign-reversed Dual Simplex pivot". Changing x_2 to $-x_2$, we change all the signs in the x_2 row of the tableau except for the 1 in the x_2 column. The ratios are $2/2 = 1$ for s_1 and $2/1 = 2$ for s_3. With the minimum ratio, s_1 enters. The next tableau (without the $-x_2$ column) is

z	x_1	x_3	s_1	s_2	s_3	rhs		
1	0	0	0	1	1	5920	=	z
0	1	0	0	1	1/2	2420	=	x_1
0	0	0	1	0	1/2	500	=	s_1
0	0	1	0	-1	0	1080	=	x_3

Thus the new optimal solution is $x_1 = 2420$, $x_3 = 1080$, $s_1 = 500$, $s_2 = s_3 = 0$.

2. [**18 pts**] *Consider the linear programming problem:*

$$\begin{array}{llllll} \text{maximize} & 4x_1 & +5x_2 & +6x_3 & -2x_4 & \\ \text{subject to} & x_1 & +x_2 & +x_3 & & \le 6 \\ & 3x_1 & & +x_3 & -x_4 & \le 8 \\ & -4x_1 & +2x_2 & & +x_4 & \le 5 \end{array}$$

$x_1, x_2, x_3, x_4 \ge 0$

In the process of solving this, we come to the basis x_3, s_2, x_4 for which

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) *Starting from this basis, perform* **ONE** *iteration of the Revised Simplex Method, obtaining the new basis, β and B^{-1}.*

$\mathbf{y}^T = [6, 0, -2]B^{-1} = [6, 0, -2]$

$$\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T \overset{\begin{matrix} x_1 & x_2 & s_1 & s_3 \end{matrix}}{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ -4 & 2 & 0 & 1 \end{pmatrix}} - \overset{\begin{matrix} x_1 & x_2 & s_1 & s_3 \end{matrix}}{\begin{pmatrix} 4 & 5 & 0 & 0 \end{pmatrix}} = \overset{\begin{matrix} x_1 & x_2 & s_1 & s_3 \end{matrix}}{\begin{pmatrix} 10 & -3 & 6 & -2 \end{pmatrix}}$$

With the most negative entry, x_2 enters. The x_2 column is $\mathbf{d} = B^{-1}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, while

$\boldsymbol{\beta} = B^{-1}\begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} = \begin{matrix} x_3 \\ s_2 \\ x_4 \end{matrix}\begin{pmatrix} 6 \\ 7 \\ 5 \end{pmatrix}$. The ratios are $6/1$, $7/1$ and $5/2$, so with the least ratio x_4 leaves. We pivot to update B^{-1} and β:

$$\left[\begin{array}{c|ccc|c} 1 & 1 & 0 & 0 & 6 \\ 1 & -1 & 1 & 1 & 7 \\ 2 & 0 & 0 & 1 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{c|ccc|c} 0 & 1 & 0 & -1/2 & 7/2 \\ 0 & -1 & 1 & 1/2 & 9/2 \\ 1 & 0 & 0 & 1/2 & 5/2 \end{array}\right]$$

Thus the new basis B^{-1} and $\boldsymbol{\beta}$ are (x_3, s_2, x_2), $B^{-1} = \begin{pmatrix} 1 & 0 & -1/2 \\ -1 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$, and $\begin{matrix} x_3 \\ s_2 \\ x_2 \end{matrix}\begin{pmatrix} 7/2 \\ 9/2 \\ 5/2 \end{pmatrix}$.

(b) *Suppose the coefficient of x_4 in the objective is changed from -2 to $-2+p$, where p is a parameter. For what interval of values of p would the basis x_3, s_2, x_4 be optimal?*

With the new c_4, we have (using the B^{-1} as given for this basis) $\mathbf{y}^T = [6, 0, -2+p]B^{-1} = [6, 0, -2+p]$ and

$$\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \overset{\begin{matrix} x_1 & x_2 & s_1 & s_3 \end{matrix}}{\begin{pmatrix} 10-4p & -3+2p & 6 & -2+p \end{pmatrix}}$$

The basis is optimal if all these entries are nonnegative, which is true for $2 \le p \le 5/2$.

(c) *For p slightly larger than the maximum of this interval, a pivot would be needed. What variables would enter and leave the basis?*

For $p > 5/2$, $\eta_1 = 10 - 4p < 0$, so x_1 would enter the basis. The x_1 column of the tableau is

$\mathbf{d} = B^{-1}\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$, while $\boldsymbol{\beta}$ is as in (a). There is only one ratio here, and x_3 leaves.

3. [**6 pts**] *What is Complementary Slackness? How can it be useful?*

Complementary Slackness is the relation between the variable values of the primal and dual problems that must hold for any optimal solutions of those problems: in each pair (x_j, η_j) and (s_i, y_i), at least one of the two must be 0. One way it is useful is in checking whether a solution is optimal. To check whether a solution of the primal problem is optimal, you first check that it is feasible and then look for a feasible solution of the dual problem that is in the relation of complementary slackness with the given solution of the primal problem.