Math 340: Answers to Assignment 9

12.2.1(a).

$$\begin{array}{ll} \text{minimize} & 50000s + 100000f \\ \text{subject to} & 5\sqrt{s} + 17\sqrt{f} \geq 40 \\ & 20\sqrt{s} + 7\sqrt{f} \geq 60 \\ & s, f \quad \geq \quad 0 \end{array}$$

(b). It violates the Proportionality, because the contribution of a variable to the left side of the constraints is proportional to the square root of the value of the variable, not the value itself. It does not violate Additivity: the contribution of a variable to the left side of a constraint does not depend on the values of the other variables.

(c). This might be more realistic because it takes into account that some women might see ads on both soap operas and football.

12.2.4. The LINGO file could be

```
model:
min =  50000 * s + 100000 * f;
5 * s^(1/2) + 17 * f^(1/2) >= 40;
20 * s^(1/2) + 7 * f^(1/2) >= 60;
end
```

The solution report is

```
Local optimal solution found at step:            18
Objective value:                               563074.4
         Variable           Value        Reduced Cost
                S        5.886590           0.0000000
                F        2.687450           0.0000000
              Row    Slack or Surplus      Dual Price
                1        563074.4            1.000000
                2       0.0000000           -15931.20
                3       0.0000000           -8148.348
```

Thus the optimal solution is to buy approximately 5.89 soap opera ads and 2.69 football ads.

12.3.6. The Hessian matrix is $\begin{pmatrix} 6x_1 & 3 \\ 3 & 2 \end{pmatrix}$. The top left entry (which is a first principal minor) is sometimes negative and sometimes positive, so this function is neither concave nor convex on $\mathbf{R}^2$.

12.4.6. The derivative $3x^2 - 6x + 2$ is 0 at $x = 1 \pm \sqrt{3}/3$. Both of these are in the interval $-2 \leq x \leq 4$. The second derivative $6x - 6$ is positive at $x = 1 + \sqrt{3}/3$ and negative at $x = 1 - \sqrt{3}/3$, so the former is a local minimum. We have $f(-2) = -25$, $f(1 + \sqrt{3}/3) \approx -1.3849$ and $f(4) = 23$, so the optimal solution, i.e. the global minimum, is at the endpoint $x = -2$.

E.1. We might start with the following patterns (of course your choice of initial patterns might be different):

Pattern 1: $1 \times 21'' + 1 \times 27'' + 1 \times 52''$ (total width $100''$, waste $0''$).

Pattern 2: $2 \times 29'' + 1 \times 37''$ (total width $95''$, waste $5''$).

The initial LINDO file is

```
max -x1 - x2
```

```
st
c21) -x1 <= -212
c27) -x1 <= -132
c29) -2 x2 <= -125
c37) -x2 <= -54
c52) -x1 <= -77
end
```

LINDO's result is

```
       OBJECTIVE FUNCTION VALUE

        1)      -274.5000

  VARIABLE        VALUE          REDUCED COST
        X1       212.000000          0.000000
        X2        62.500000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
      C21)         0.000000          1.000000
      C27)        80.000000          0.000000
      C29)         0.000000          0.500000
      C37)         8.500000          0.000000
      C52)       135.000000          0.000000
```

Thus the shadow prices are 1 for 21″ finals, 0.5 for 29″ finals, 0 otherwise. To find a pattern that could enter, we can use the following LINDO file for integer linear programming:

```
max p21 + 0.5 p29
st
21 p21 + 27 p27 + 29 p29 + 37 p37 + 52 p52 <= 100
end
gin 5
```

A solution with objective value greater than 1 represents a pattern that would enter the basis. The optimal solution, with objective value 4, has $p_{21} = 4$ and all others 0, i.e. the pattern

Pattern 3: $4 \times 21''$ (total width $84''$, waste $16''$).

Adding the new variable x_3 in to the first problem file, with coefficients -1 in the objective and -4 in constraint `c21`, the new solution has objective value -214.5, with shadow prices 0.25 for $21''$, 0.75 for $27''$ and 0.5 for $29''$. Changing the objective in the integer linear programming problem to `0.25 p21 + 0.75 p27 + 0.5 p29`, we get a solution with objective value 2.25: $p_{27} = 3$, all others 0, i.e. the pattern

Pattern 4: $3 \times 27''$ (total width $81''$, waste $19''$).

Adding in x_4 to the first problem file, with coefficients -1 in the objective and -3 in `c27`, the new solution has objective value -191.5833, with shadow prices 0.25 for $21''$, 0.333333 for $27''$, 0.5 for $29''$, 0 for $37''$ and 0.416667 for $52''$. Changing the objective accordingly in the integer linear programming problem, we get a solution with objective value 1.5: $p_{29} = 3$, all others 0, i.e.

Pattern 5: $3 \times 29''$ (total width $87''$, waste $13''$).

Adding in x_5 to the first problem file, the new solution has objective value -188.75, with shadow prices 0.25 for $21''$, 0.333333 for $27''$, $29''$ and $37''$, and 0.416667 for $52''$. Changing the objective accordingly in the integer linear programming problem, we get a solution with objective value 1.166666: $p_{21} = 2$, $p_{27} = 2$, all others 0, i.e.

Pattern 6: $2 \times 21'' + 2 \times 27''$ (total width $96''$, waste $4''$).

Adding in x_6 to the first problem file, the new solution has objective value -184.1667, with shadow prices 0.25 for $21''$ and $27''$, 0.333333 for $29''$ and $37''$, 0.5 for $52''$. Changing the objective accordingly in the integer linear programming problem, we get a solution with objective value 1.166666: $p_{21} = 2$, $p_{29} = 2$, all others 0, i.e.

Pattern 7: $2 \times 21'' + 2 \times 29''$ (total width $100''$, waste 0).

Adding in x_7 to the first problem file, the new solution has objective value -182.75, with shadow prices 0.25 for $21''$, $27''$ and $29''$ and 0.5 for $37''$ and $52''$. Changing the objective accordingly in the integer linear programming problem, we get a solution with objective value 1.25: $p_{21} = 1$, $p_{37} = 2$, all others 0, i.e.

Pattern 8: $1 \times 21'' + 2 \times 37''$ (total width $95''$, waste $5''$).

Adding in x_8 to the first problem file, the new solution has objective value -179.6, with shadow prices 0.2, 0.3, 0.3, 0.4 and 0.5 for $21''$, $27''$, $29''$, $37''$ and $52''$ respectively. Changing the objective accordingly in the integer linear programming problem, the optimal solution has objective value 1. Thus there are no new patterns that could enter the basis. The optimal solution for 179.6 raw rolls uses 77 of Pattern 1, 28.8 of Pattern 2, 27.5 of Pattern 6, 33.7 of Pattern 7 and 12.6 of Pattern 8.

Of course this is not an integer solution. However, if we ask for an integer solution by adding

```
gin 8
```

at the end of the file, LINDO provides a solution using 180 raw rolls: 77 of Pattern 1, 26 of Pattern 2, 1 of Pattern 4, 1 of Pattern 5, 26 of Pattern 6, 35 of Pattern 7 and 14 of Pattern 8. This satisfies all requirements, with one extra $21''$ final roll. Since according to the linear programming solution it's impossible to get by with 179 raw rolls, this is the best possible.

You may not get the same solution, because there may be other optimal solutions, but they should all involve the same objective value.

E.2(a). From $\partial f/\partial x = -400x(y - x^2) - 2 + 2x = 0$ and $\partial f/\partial y = 200(y - x^2) = 0$ we get $y = x^2$ and $x = 1$. The only critical point is $(1, 1)$. Now $f(1, 1) = 0$ while $f(x, y) \geq 0$ everywhere, so clearly this is a global minimum.

(b). The Hessian $H = \begin{pmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200 \end{pmatrix}$. The principal minors are $1200x^2 - 400y + 2$, 200 and $80000x^2 - 80000y + 400 = 400(200x^2 - 200y + 1)$. Note that $1200x^2 - 400y + 2 \geq 2(200x^2 - 200y + 1)$. Thus the matrix is positive semidefinite if $y \leq x^2 + 1/200$. However, $\{(x, y) : y \leq x^2 + 1/200\}$ is not a convex set. A convex set on which f is convex would be $\{(x, y) : y \leq a + bx\}$ where the line $y = a + bx$ is tangent to the parabola $y = x^2 + 1/200$. The tangent line at $x = x_0$ is $y = x_0^2 + 1/200 + 2x_0(x - x_0) = 2x_0 x - x_0^2 + 1/200$, so the answer is $\{(x, y) : y \leq 2x_0 x - x_0^2 + 1/200\}$ for any real number x_0.