Math 340: Answers to Assignment 8

10.1.1. The initial basis consists of the slack variables s_1, s_2, s_3, $B^{-1} = I$, and $\boldsymbol{\beta} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$.

First iteration: $\mathbf{y}^T = [0,0,0]B^{-1} = [0,0,0]$, $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{matrix} x1 & x2 & x3 \\ (-3 & -1 & -1) \end{matrix}$. x_1 enters.

$$\mathbf{d} = B^{-1} \overset{x_1}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{ ratios } \begin{matrix} 6 \\ 2 \\ --- \end{matrix} \quad \Leftarrow. \ s_2 \text{ leaves.}$$

$$\left[\begin{array}{c|ccc|c} 1 & 1 & 0 & 0 & 6 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc|c} 0 & 1 & -1/2 & 0 & 4 \\ 1 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \text{ so } B^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \beta = \begin{matrix} s_1 \\ x_1 \\ s_3 \end{matrix} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

Second iteration: $\mathbf{y}^T = [0,3,0]B^{-1} = [0,3/2,0]$, $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{matrix} x2 & x3 & s_2 \\ (-1 & -5/2 & 3/2) \end{matrix}$. x_3 enters.

$$\mathbf{d} = B^{-1} \overset{x_3}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \end{pmatrix}, \text{ ratios } \begin{matrix} 8/3 \\ --- \\ 2 \end{matrix} \begin{matrix} \\ \\ \Leftarrow \end{matrix} . \ s_3 \text{ leaves.}$$

$$\left[\begin{array}{c|ccc|c} 3/2 & 1 & -1/2 & 0 & 4 \\ -1/2 & 0 & 1/2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc|c} 0 & 1 & -1/2 & -3/2 & 1 \\ 0 & 0 & 1/2 & 1/2 & 3 \\ 1 & 0 & 0 & 1 & 2 \end{array}\right] \text{ so } B^{-1} = \begin{pmatrix} 1 & -1/2 & -3/2 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta = \begin{matrix} s_1 \\ x_1 \\ x_3 \end{matrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Third iteration: $\mathbf{y}^T = [0,3,1]B^{-1} = [0,3/2,5/2]$, $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{matrix} x2 & s_2 & s_3 \\ (3/2 & 3/2 & 5/2) \end{matrix}$.
The current solution is optimal: $x_1 = 3$, $x_2 = 0$, $x_3 = 2$, $z = \mathbf{y}^T\mathbf{b} = 11$.

E.1. I'll begin by reconstructing the tableau for the original problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	1	0	0	1	2	0	220	=	z
0	4/3	1	0	2/3	−1/3	0	30	=	x_2
0	1/3	0	1	−1/3	2/3	0	20	=	x_3
0	−1/3	0	0	1/3	−2/3	1	10	=	s_3

(a). If $b_2 = p$, this becomes

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	1	0	0	1	2	0	$80 + 2p$	=	z
0	4/3	1	0	2/3	−1/3	0	$160/3 - 1/3\,p$	=	x_2
0	1/3	0	1	−1/3	2/3	0	$-80/3 + 2/3\,p$	=	x_3
0	−1/3	0	0	1/3	−2/3	1	$170/3 - 2/3\,p$	=	s_3

This is optimal if $40 \leq p \leq 85$, and in this interval $z = 80 + 2p$. If $p < 40$, x_3 must leave; in a Dual Simplex pivot s_1 enters.

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	2	0	3	0	4	0	$4p$	$=$	z
0	2	1	2	0	1	0	p	$=$	x_2
0	-1	0	-3	1	-2	0	$80 - 2p$	$=$	s_1
0	0	0	1	0	0	1	30	$=$	s_3

This is optimal if $0 \leq p \leq 40$. In this interval, $z = 4p$. If $p < 0$, x_2 must leave, but there is nothing to enter, so the problem would be infeasible.

Returning to the first tableau, if $p > 85$, s_3 leaves; ratios are 3 for both x_1 and s_2. We let x_1 enter:

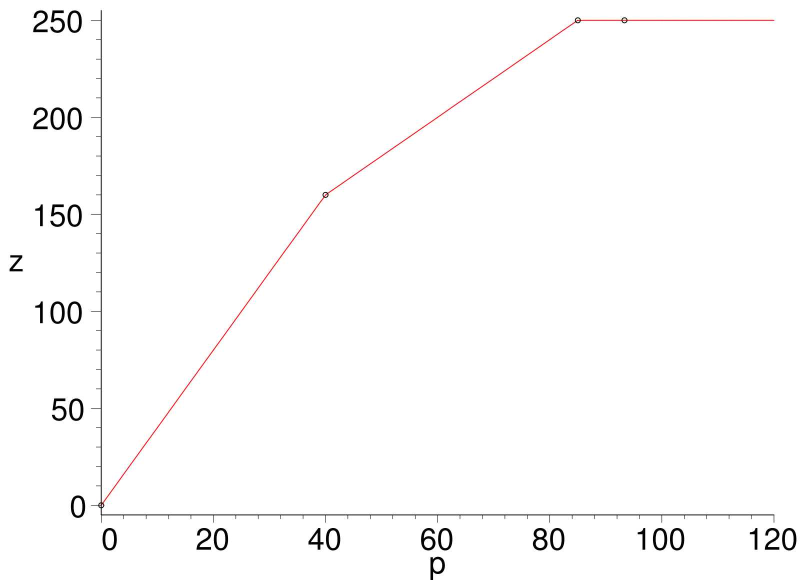
z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	0	0	2	0	3	250	$=$	z
0	0	1	0	2	-3	4	$280 - 3p$	$=$	x_2
0	0	0	1	0	0	1	30	$=$	x_3
0	1	0	0	-1	2	-3	$-170 + 2p$	$=$	x_1

This is optimal if $85 \leq p \leq 280/3 = 93\,1/3$, and in this interval $z = 250$. For $p > 280/3$, x_2 must leave, and s_2 enters:

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	0	0	2	0	3	250	$=$	z
0	0	$-1/3$	0	$-2/3$	1	$-4/3$	$-280/3 + p$	$=$	s_2
0	0	0	1	0	0	1	30	$=$	x_3
0	1	$2/3$	0	$1/3$	0	$-1/3$	$50/3$	$=$	x_1

This is optimal for $p \geq 280/3$, and $z = 250$ in this interval.

The graph of the optimal z as a function of p looks like this:



(b). The additional constraint is $x_1 + x_2 + x_3 \leq 40$ or $x_1 + x_2 + x_3 + s_4 = 40$. Using the tableau for the basis x_2, x_3, s_3, we substitute in for the basic variables x_2 and x_3 to get a new row for that tableau. The tableau becomes

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs		
1	1	0	0	1	2	0	0	220	=	z
0	4/3	1	0	2/3	−1/3	0	0	30	=	x_2
0	1/3	0	1	−1/3	2/3	0	0	20	=	x_3
0	−1/3	0	0	1/3	−2/3	1	0	10	=	s_3
0	−2/3	0	0	−1/3	−1/3	0	1	−10	=	s_4

Because of the -10 in the rhs, this is not feasible. We need a Dual Simplex pivot, where s_4 leaves the basis. The ratios are 3/2 for x_1, 3 for s_1, 6 for s_2, so x_1 enters. The next tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs		
1	0	0	0	1/2	3/2	0	3/2	205	=	z
0	0	1	0	0	−1	0	2	10	=	x_2
0	0	0	1	−1/2	1/2	0	1/2	15	=	x_3
0	0	0	0	1/2	−1/2	1	−1/2	15	=	s_3
0	1	0	0	1/2	1/2	0	−3/2	15	=	x_1

This is optimal: $x_1 = 15$, $x_2 = 10$, $x_3 = 15$, $s_1 = s_2 = s_4 = 0$, $s_3 = 15$, $z = 205$.

E.2. We begin with basis s_1, s_2, $B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{\beta} = \mathbf{b} = \begin{pmatrix} 8 \\ 21 \end{pmatrix}$;

$$\mathbf{y}^T = [0,0]B^{-1} = [0,0],\ \boldsymbol{\eta}^T_{NBV} = \mathbf{y}^T N - \mathbf{c}^T_{NBV} = \begin{matrix} x1 & x2 & x3 & x4 \\ (-13 & -10 & -12 & -17) \end{matrix}.$$ x_4 enters.

$$\mathbf{d} = B^{-1} \overset{x_4}{\begin{pmatrix} 2 \\ 5 \end{pmatrix}} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \text{ ratios } \begin{matrix} 4 & \Leftarrow \\ 21/5 & \end{matrix}. \ s_1 \text{ leaves.}$$

$$\left[\begin{array}{c|cc|c} 2 & 1 & 0 & 8 \\ 5 & 0 & 1 & 21 \end{array}\right] \longrightarrow \left[\begin{array}{c|cc|c} 1 & 1/2 & 0 & 4 \\ 0 & -5/2 & 1 & 1 \end{array}\right] \text{ so } B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -5/2 & 1 \end{pmatrix}, \ \beta = \begin{matrix} x_4 \\ s_2 \end{matrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\mathbf{y}^T = [17, 0]B^{-1} = [17/2, 0], \ \boldsymbol{\eta}^T_{NBV} = \mathbf{y}^T A_N - \mathbf{c}^T_{NBV} = \begin{matrix} x1 & x2 & x3 & s_1 \\ \big(25/2 & -3/2 & 5 & 17/2 \big) \end{matrix}. \ x_2 \text{ enters.}$$

$$\mathbf{d} = B^{-1} \overset{x_2}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \text{ ratios } \begin{matrix} 8 & \\ 2 & \Leftarrow \end{matrix}. \ s_2 \text{ leaves.}$$

$$\left[\begin{array}{c|cc|c} 1/2 & 1/2 & 0 & 4 \\ 1/2 & -5/2 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{c|cc|c} 0 & 3 & -1 & 3 \\ 1 & -5 & 2 & 2 \end{array}\right] \text{ so } B^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}, \ \beta = \begin{matrix} x_4 \\ x_2 \end{matrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{y}^T = [17, 10]B^{-1} = [1, 3], \ \boldsymbol{\eta}^T_{NBV} = \mathbf{y}^T N - \mathbf{c}^T_{NBV} = \begin{matrix} x1 & x3 & s_1 & s_2 \\ \big(2 & 2 & 1 & 3 \big) \end{matrix}.$$

Optimal solution: $x_1 = 0$, $x_2 = 2$, $x_3 = 0$, $x_4 = 3$, $z = \mathbf{y}^T\mathbf{b} = 71$.

E.3. Note that with the Professor's basis, $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 10 \\ 1 \end{pmatrix}$ which is feasible, so we are in Phase II (the Professor has already done Phase I).

$$\mathbf{y}^T = [1, 2, 0]B^{-1} = [0, -1, 0], \ \boldsymbol{\eta}^T_{NBV} = \mathbf{y}^T N - \mathbf{c}^T_{NBV} = \begin{matrix} x3 & x4 & s_2 & s_3 \\ \big(-2 & -1 & -1 & 0 \big) \end{matrix}. \ x_3 \text{ enters.}$$

$$\mathbf{d} = B^{-1} \overset{x_3}{\begin{pmatrix} 60 \\ -4 \\ 3 \end{pmatrix}} = \begin{pmatrix} -14 \\ 9 \\ 1 \end{pmatrix}, \text{ ratios } \begin{matrix} --- & \\ 10/9 & \\ 1 & \Leftarrow \end{matrix}. \ s_1 \text{ leaves.}$$

$$\left[\begin{array}{c|ccc|c} -14 & 0 & 5 & 2 & 1 \\ 9 & 0 & -3 & -1 & 10 \\ 1 & 1 & 20 & 7 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc|c} 0 & 14 & 285 & 100 & 15 \\ 0 & -9 & -183 & -64 & 1 \\ 1 & 1 & 20 & 7 & 1 \end{array}\right] \text{ so } B^{-1} = \begin{pmatrix} 14 & 285 & 100 \\ -9 & -183 & -64 \\ 1 & 20 & 7 \end{pmatrix},$$

$$\beta = \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \begin{pmatrix} 15 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{y}^T = [1, 2, 6]B^{-1} = [2, 39, 14], \ \boldsymbol{\eta}^T_{NBV} = \mathbf{y}^T N - \mathbf{c}^T_{NBV} = \begin{matrix} x4 & s_1 & s_2 & s_3 \\ \big(69 & 2 & 39 & 14 \big) \end{matrix}.$$

Nothing to enter, so this is the optimal solution: $x_1 = 15$, $x_2 = x_3 = 1$, $x_4 = 0$, $z = 23$.