Math 340: Answers to Assignment 7

6.8.2(a). The optimal tableau for Sugarco is in Table 11 on p. 288. The shadow prices for the constraints are the entries in the objective row for the slack variables s_1 and s_2: $y_1 = 4$, $y_2 = 1$.

(b). With $B^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$, adding 10 to b_1 adds $\begin{pmatrix} 15 \\ -5 \end{pmatrix}$ to $\boldsymbol{\beta}$, making the basic solution $x_3 = 40$, $x_2 = 20$. This is still feasible, so the optimal basis doesn't change. The increase in z is $10y_1 = 40$, so the profit becomes 340 ¢.

(c). This time we're decreasing b_1 by 10; again the basic solution is feasible, and the optimal basis doesn't change. The decrease in z is 40, so the profit becomes 260 ¢.

(d). This time we decrease b_1 by 20, adding $\begin{pmatrix} -30 \\ 10 \end{pmatrix}$ to $\boldsymbol{\beta}$ and making the basic solution $x_3 = -5$, $x_2 = 35$. The z value in the basic solution is $300 - 20y_1 = 220$. The basic solution is not feasible, so we do a Dual Simplex pivot. From the tableau

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	3	0	0	4	1	220	=	z
0	1/2	0	1	3/2	−1/2	−5	=	x_3
0	1/2	1	0	−1/2	1/2	35	=	x_2

x_3 leaves, and s_2 enters (with the only negative entry in the x_3 row). The next tableau is

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	4	0	2	7	0	210	=	z
0	−1	0	−2	−3	1	10	=	s_2
0	1	1	1	1	0	30	=	x_2

which is optimal. The profit in the optimal solution is 210 ¢.

6.8.7(a). In problem 6.3.8(b) (assignment 5) we found that the current basis would be optimal if the required HIW exposure is between 4 million and 84 million. In this case with (in my version of the problem) $\mathbf{b} = \begin{pmatrix} -40 \\ -24 \end{pmatrix}$ and $\mathbf{y}^T = (5, 7.5)$, my z value is $\mathbf{y}^T\mathbf{b} = -380$, i.e. the cost is \$380,000.

(b). With $\mathbf{b} = \begin{pmatrix} -28 \\ -20 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} -.15 & .025 \\ .025 & -.0875 \end{pmatrix} \begin{pmatrix} -28 \\ -20 \end{pmatrix} = \begin{pmatrix} 3.7 \\ 1.05 \end{pmatrix}$ which is feasible and thus optimal. The increase of 4 in b_2 increases my version of the objective by $7.5 \times 4 = 30$ to -290, i.e. the cost is \$290,000.

6.8.9. In the Dakota problem (page 276), the allowable increase for b_2 (the finishing hours) is 4 and the allowable increase for b_3 (the carpentry hours) is 2. In the 100% rule, $\sum r_j = 2/4 + 1/2 = 1$, so the basis will still be optimal (or of course you could just calculate $B^{-1}\mathbf{b} = \begin{pmatrix} 20 \\ 8 \\ 2.5 \end{pmatrix}$). The shadow prices being 0, 10 and 10, the increase in z is $10 \times 2 + 10 \times 1 = 30$, so the new optimal z value is 310.

6.11.1. The initial tableau (after putting the constraints into $\le$ form) is

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	2	0	1	0	0	0	=	z
0	−1	−1	1	1	0	−5	=	s_1
0	−1	2	−4	0	1	−8	=	s_2

This is suitable for the Dual Simplex method since the initial basic solution is feasible for the dual but not the primal. s_2 leaves, ratios are 2/1 for x_1 and 1/4 for x_3, so x_3 enters.

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	7/4	1/2	0	0	1/4	-2	=	z
0	$-5/4$	$-1/2$	0	1	1/4	-7	=	s_1
0	1/4	$-1/2$	1	0	$-1/4$	2	=	x_3

Now s_1 leaves, ratios are 7/5 for x_1 and 1/1 for x_2, so x_2 enters.

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	1/2	0	0	1	1/2	-9	=	z
0	5/2	1	0	-2	$-1/2$	14	=	x_2
0	3/2	0	1	-1	$-1/2$	9	=	x_3

This is optimal: $x_1 = 0$, $x_2 = 14$, $x_3 = 9$, $z = -9$.

6.11.3. With the new $b_1 = 20$ instead of 48, $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} -4 \\ 8 \\ 2 \end{pmatrix}$, and the z value is $\mathbf{y}^T\mathbf{b} = 280$.

Thus the initial tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	5	0	0	10	10	280	=	z
0	0	-2	0	1	2	-8	-4	=	s_1
0	0	-2	1	0	2	-4	8	=	x_3
0	1	1.25	0	0	$-.5$	1.5	2	=	x_1

We use the Dual Simplex method: s_1 leaves, the ratios are 5/2 for x_2 and 10/8 for s_3, so s_3 enters. The next tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	2.5	0	1.25	12.5	0	275	=	z
0	0	.25	0	$-.125$	$-.25$	1	0.5	=	s_3
0	0	-1	1	$-.5$	1	0	10	=	x_3
0	1	.875	0	.1875	$-.125$	0	1.25	=	x_1

This is optimal: $x_1 = 1.25$, $x_2 = 0$, $x_3 = 10$, $z = 275$.

E.1. Note that $x_2 = -1$ is not a problem because x_2 is unrestricted in sign (URS). Plugging the given solution in to the constraints, we get $s_1 = 2$, $s_2 = 0$, $s_3 = 0$ so this is feasible for the primal. By Complementary Slackness we must have $\eta_1 = \eta_2 = \eta_4 = y_1 = 0$. Note also that y_3 is URS since the third constraint is an inequality. The equations of the dual now say

$$\begin{aligned} 3y_2 &= 6 \\ y_2 + y_3 &= 1 \\ -y_2 + y_3 - \eta_3 &= -1 \\ y_3 &= -1 \end{aligned}$$

The first equation says $y_2 = 2$, the fourth says $y_3 = -1$, the second is satsified, but the third (with $y_2 = 2$ and $y_3 = -1$) says $\eta_3 = -2$. This is not feasible (x_3 is an ordinary "≥ 0" variable, and thus so is η_3). So the solution is not optimal.

E.2(a). Using the proposed solution of the primal, we get $s_1 = b_1 - 6 - 4a_{13}$ (which must be 0 since this constraint is an equality) and $s_2 = 2$. Using the proposed solution of the dual, we get $\eta_1 = -1 - c_1$, $\eta_2 = -a_{12} - 2$, $\eta_3 = -a_{13} - 1$. By complementary slackness, η_1 and η_3 must be 0, while η_2 can be anything ≥ 0; y_1 is allowed to be negative since the first constraint is an equality, and y_2 must be 0 (which it is). Thus $c_1 = -1$, $a_{13} = -1$, $a_{12} \leq 2$, and $b_1 = 6 + 4a_{13} = 2$.
(b). Any student using the Simplex Method will find an optimal solution which is basic. Since P has two constraints, a basic solution of P has only two basic variables, and thus at most two variables can be nonzero. But the Professor's solution has three nonzero variables: x_1, x_3 and s_2. Thus P does not have a unique optimal solution, and since the Professor's solution for P is not a basic solution no student will obtain it. They will obtain his solution for the dual D: by complementary slackness, any optimal solution of D must have $\eta_1 = \eta_3 = y_2 = 0$, and $y_1 = c_1 = -1$.

E.3. If $y_2 = 3$ in an optimal solution of the dual, complementary slackness says any optimal solution of the primal must have $s_2 = 0$, i.e. $9x_1 - 6x_2 = 6$, or $x_1 = (2/3)(x_2 + 1)$. Substituting this in, the first equation of the primal then says $4 + x_3 + s_1 = 4$, or $x_3 + s_1 = 0$. Since $x_3 \geq 0$ and $s_1 \geq 0$, the only way this can happen is $x_3 = s_1 = 0$. And then the third equation of the primal says $-2 + s_3 = -1$, or $s_3 = 1$.

Now by complementary slackness we must have $y_3 = 0$, and (since x_1 and x_2 are URS) $\eta_1 = \eta_2 = 0$. The equations of the dual then say

$$\begin{aligned} 6y_1 + 27 + 0 &= c_1 \\ -4y_1 - 18 + 0 &= -34 \\ 3y_1 - \eta_3 &= c_3 \end{aligned}$$

From the second equation, $y_1 = (-34 + 18)/(-4) = 4$. Then the first and third say $c_1 = 51$ and $c_3 = 12 - \eta_3$. The optimal objective value can be calculated as $4y_1 + 6y_2 - y_3 = 34$. The only thing we can't determine about the problem is c_3: all we can say there is $c_3 \leq 12$ since $\eta_3 \geq 0$.