Math 340: Answers to Assignment 5

6.3.3. The Dakota problem has $B^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}$ so with the new $\mathbf{b} = \begin{pmatrix} b_1 \\ 20 \\ 8 \end{pmatrix}$ we have $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} b_1 - 24 \\ 8 \\ 2 \end{pmatrix}$. This is optimal as long as it is feasible, i.e. for $b_1 \ge 24$. For $b_1 = 30$, $\boldsymbol{\beta} = \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$, i.e. $s_1 = 6$, $x_3 = 8$, $x_1 = 2$.

6.3.6(a). In the optimal solution of the original problem, x_1 is nonbasic, with reduced cost 3. This means that each Type I candy bar made reduces z by 3 cents. The current basis remains optimal if the profit c_1 for Type I candy bars increases by no more than 3 cents, i.e. if $c_1 \le 6$. If the profit were 7 cents, the tableau would be

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	-1	0	0	4	1	300	=	z
0	1/2	0	1	3/2	$-1/2$	25	=	x_3
0	1/2	1	0	$-1/2$	1/2	25	=	x_2

x_1 enters; there is a tie for leaving variable, so I'll choose x_3 (the higher row in the tableau) to leave. The next tableau is

z	x_1	x_2	x_3	s_1	s_2	rhs		
1	0	0	2	7	0	350	=	z
0	1	0	2	3	-1	50	=	x_1
0	0	1	-1	-2	1	0	=	x_2

This is optimal. The new optimal solution has $x_1 = 50$, $x_2 = x_3 = s_1 = s_2 = 0$, $z = 350$.

(b). With $\mathbf{c}_{BV}^T = [5, c_2]$ have $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = [15/2 - c_2/2, -5/2 + c_2/2]$ and $\eta_1 = \mathbf{y}^T A_1 - 3 = -1/2 + c_2/2$. The current solution is optimal if all these are nonnegative, i.e. $5 \le c_2 \le 15$. At a profit of 13 cents, the current solution would still be optimal, with the same values $x_2 = x_3 = 25$, and $z = 13 \times 25 + 5 \times 25 = 450$ cents.

(c). With b_1 changed from 50 to t, $\beta = B^{-1}\begin{pmatrix} t \\ 100 \end{pmatrix} = \begin{pmatrix} 3t/2 - 50 \\ 50 - t/2 \end{pmatrix}$. Thus the current basis remains feasible (and therefore optimal) if $100/3 \le t \le 100$.

(d). Within the above interval, the shadow price of the sugar constraint is 4 cent per oz. The additional 10 oz of sugar raises the profit by 40 cents, to 340 cents. We have $\beta = \begin{pmatrix} 40 \\ 20 \end{pmatrix}$, i.e. they should produce 20 type 1 and 40 type 2 candy bars. If only 30 oz of sugar were available, this would be outside the interval where the current basis is feasible, and the questions can't be answered (except by solving the new problem from scratch or using the Dual Simplex Method, which we didn't have yet).

(e). The reduced cost of the new type 1 bar would be $.5 \times 4 + .5 \times 1 - 3 = -.5 < 0$, so it would become profitable to produce these.

(f). The reduced cost of the type 4 bar would be $3 \times 4 + 4 \times 1 - 17 = -1 < 0$, so it would be profitable to produce these.

6.3.8(a). In my version of the problem with $\mathbf{c}_{BV}^T = [c_1, -100]$ we have $\mathbf{y}^T = [-5/2 - 3/20c_1, 35/4 + 1/40c_1]$. The current basis is optimal if $-350 \le c_1 \le -50/3$, i.e. if the cost of a comedy ad is between \$16,666.67 and \$350,000.

(b). With b_1 changed from -28 to $-t$, we have $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} (3t-12)/20 \\ (84-t)/40 \end{pmatrix}$. The current basis is optimal as long as $4 \le t \le 84$, i.e. the required HIW exposures are from 4 million to 84 million. A requirement of 40 million (i.e. $t = 40$) is in this interval, so the optimal solution would be $x_1 = 5.4$, $x_2 = 1.1$.

(c). Pricing it out: $\mathbf{y}^T A_{\text{news}} - c_{\text{news}} = [5, 7.5]\begin{pmatrix} -12 \\ -7 \end{pmatrix} - (-110) = -2.5 < 0$. Therefore Dorian should advertise on the news program.

6.3.9. The new $\mathbf{b}$ is the old $\mathbf{b} + \Delta\mathbf{e}$ where $\mathbf{e}$ is the vector with $e_i = 1$ and all other entries 0. So the new $\boldsymbol{\beta}$ is the old $\boldsymbol{\beta} + \Delta B^{-1}\mathbf{e}$, and $B^{-1}\mathbf{e}$ is column number i of B^{-1}.

6.4.7. In the Dakota problem the desks and chairs are basic, so we use the 100% rule. The "OBJ COEFFICIENT RANGES" in the LINDO output on p. 282 shows that a price of \$65 for desks, a \$5 increase, is 25% of the allowable increase; \$25 for tables, a \$5 decrease, is 0% of the allowable decrease; \$18 for chairs, a \$2 decrease, is 40% of the allowable decrease. The total is 65% < 100%. Therefore the current basis would remain optimal. The new z value would be $65 \times 2 + 18 \times 8 = \$274$.

6.4.13. Note that

$$\begin{aligned} r_1[U_1, b_2] + r_2[b_1, L_2] + (1 - r_1 - r_2)[b_1, b_2] &= [r_1(U_1 - b_1) + b_1, r_2(L_2 - b_2) + b_2] \\ &= [b_1 + \Delta b_1, b_2 + \Delta b_2] = [b_1', b_2'] \end{aligned}$$

Now $B^{-1}\begin{pmatrix} b_1' \\ b_2' \end{pmatrix} = r_1 B^{-1}\begin{pmatrix} U_1 \\ b_2 \end{pmatrix} + r_2 B^{-1}\begin{pmatrix} b_1 \\ L_2 \end{pmatrix} + (1 - r_1 - r_2)B^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Note that $B^{-1}\begin{pmatrix} U_1 \\ b_2 \end{pmatrix} \ge 0$ because the current basis is still optimal if b_1 is changed to U_1 with b_2 unchanged. Similarly, $B^{-1}\begin{pmatrix} b_1 \\ L_2 \end{pmatrix} \ge 0$, while $B^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \ge 0$ because the current basis is optimal with b_1 and b_2 unchanged. Thus if $r_1 + r_2 \le 1$, $B^{-1}\begin{pmatrix} b_1' \\ b_2' \end{pmatrix}$ is a linear combination of nonnegative vectors with nonnegative coefficients, and therefore is nonnegative.

E.1(a). We have $\mathbf{b} = \begin{pmatrix} 15 \\ 18 \\ 4 \end{pmatrix}$ so $\beta = B^{-1}\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, i.e. the optimal solution is $x_1 = 2$, $x_2 = 1$, $x_3 = x_4 = 0$. We have $\mathbf{c}_{BV}^T = [-1, 0, 4]$ so the vector of dual prices is $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = [0, -1/3, 2]$ (the fact that $y_2 < 0$ is no problem because the second constraint is an equality). The vector of reduced costs is $\boldsymbol{\eta}^T = \mathbf{y}^T A - \mathbf{c}^T = [0, 0, 2, 7]$.

(b). The reduced cost of x_5 would be $\eta_5 = \mathbf{y}^T\begin{pmatrix} 1 \\ 3 \\ p \end{pmatrix} - 3 = 2p - 4$, so the solution is optimal if $p \ge 2$. With $p = 2$, the current solution is still optimal but is not the only optimal solution: x_5 could enter the basis. The x_5 column of the tableau would be $B^{-1}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1/2 \end{pmatrix}$. Thus $x_1 = 2 + x_5$, $s_1 = 1$ and $x_2 = 1 - x_5/2$. The other optimal basic solution (when $x_5 = 2$) is $x_1 = 4$, $x_2 = x_3 = x_4 = 0$, $x_5 = 2$.