Math 340: Answers to Assignment 4

6.1.3. Note that "the weekly demand for soldiers is at least 20" means that the demand constraint in the Giapetto problem is $x_1 \le K$ where $K \ge 20$. It doesn't mean that the constraint is $x_1 \ge 20$. Also, Figure 1 on Page 263 should look like this:

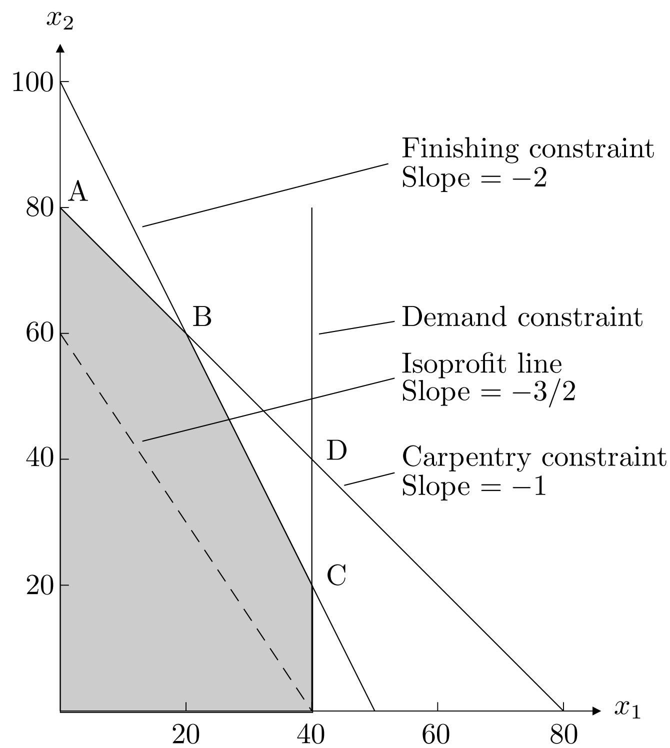


Figure 1

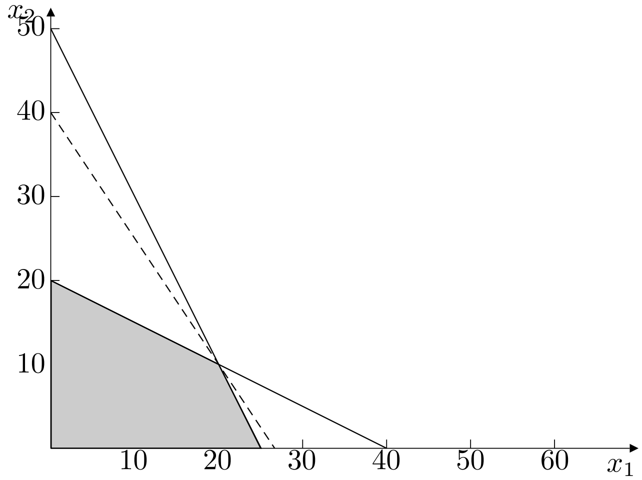
If $K \ge 20$, point B, which is the intersection of the lines for the finishing and carpentry constraints, is still in the feasible region. The isoprofit line through B intersects the feasible region only at B, and no isoprofit line for higher profit can intersect the feasible region. Therefore the optimal solution remains the same.

6.1.5. Note that Table 1 has a misprint: Radio 2 should require 1 hour of Laborer 2, not 2. Thus the second constraint (for Laborer 2) is $2x_1 + x_2 \le 50$, and Radio 2's contribution to the profit is $22 - 2 \times 5 - 1 \times 6 - 4 = 2$ dollars. The feasible region is shown below. The dotted line is the isoprofit line for $z = 80$, which goes through the optimal solution $x_1 = 20$, $x_2 = 10$.

(a). The basis in question is x_1, x_2, since these are the variables that have nonzero values in the optimal solution. The given solution is optimal as long as the slope of the isoprofit line is between $-1/2$ (the slope of the line $x_1 + 2x_2 = 40$) and -2 (the slope of the line $2x_1 + x_2 = 50$) inclusive. If the coefficient c_1 of x_1 in the objective changes, while c_2 stays at 2, the slope of the isoprofit line is $-c_1/2$, so you need $-2 \le -c_1/2 \le -1/2$, i.e. $1 \le c_1 \le 4$. Now if the price of the Type 1 radio is p_1, $c_1 = p_1 - 22$. So the answer is that the price could be from \$23 to \$26.

(b). If c_2 changes while c_1 stays at 3, the slope of the isoprofit line is $-3/c_2$, so you need $-2 \le -3/c_2 \le -1/2$, i.e. $3/2 \le c_2 \le 6$. If the price of the Type 2 radio is p_2, $c_2 = p_2 - 20$. Thus the price could be from \$21.50 to \$26.

(c). If the right side of the first constraint was changed from 40 to 30, the lines $x_1 + 2x_2 = 30$ and $2x_1 + x_2 = 50$ would still intersect in the first quadrant, and because the slopes would still be the same this point would still be the optimal solution: $x_1 = 70/3$, $x_2 = 10/3$, with $z = 230/3$.



Feasible region for 6.1.5

(d). If the right side of the second constraint was changed from 50 to 60, the lines would again intersect in the first quadrant, again giving the optimal solution: $x_1 = 80/3$, $x_2 = 20/3$, with $z = 280/3$.

(e). In (c), a decrease of 10 in b_1 caused a decrease of 10/3 in z, keeping the same basis, so the shadow price for the first constraint is 1/3. In (d), an increase of 10 in b_2 caused an increase of 40/3 in z, keeping the same basis, so the shadow price for the second constraint is 4/3.

6.2.2. We have $B = \begin{array}{c} \begin{array}{cc} x_2 & s_1 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$ so $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

$B^{-1}N = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{array}{c} \begin{array}{cc} x_1 & s_2 \end{array} \\ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} x_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$. These columns go in the main part of the tableau under x_1 and s_2.

$\mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ so $\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ which goes in the main part of the rhs column.

$\mathbf{c}_{BV}^T = \begin{array}{c} \begin{array}{cc} x_2 & s_1 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$ and $\mathbf{c}_{NBV}^T = \begin{array}{c} \begin{array}{cc} x_1 & s_2 \end{array} \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \end{array}$ so so $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{array}{c} \begin{array}{cc} x_1 & s_2 \end{array} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \end{array}$ which goes in the objective row under x_1 and s_2.

$z^* = \mathbf{c}_{BV}^T \boldsymbol{\beta} = 2$ which goes in the rhs column of the objective row.

Thus the tableau is

z	x_1	x_2	s_1	s_2	rhs		
1	2	0	0	1	2	=	z
0	1	1	0	1	2	=	x_2
0	1	0	1	-1	2	=	s_1

6.3.1. The Dakota problem has $B^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}$ with basic variables s_1, x_3, x_1 (see pages 268 and 270 in the text). We are to take $\mathbf{c}^T = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 60 & 30 & c_3 & 0 & 0 & 0 \end{pmatrix} \end{array}$, so

$\mathbf{c}_{BV}^T = \begin{array}{c} \begin{array}{ccc} s_1 & x_3 & x_1 \end{array} \\ \begin{pmatrix} 0 & c_3 & 60 \end{pmatrix} \end{array}$. Then $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = \begin{pmatrix} 0 & -30+2c_3 & 90-4c_3 \end{pmatrix}$ and $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{array}{c} \begin{array}{ccc} x_2 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 45-2c_3 & -30+2c_3 & 90-4c_3 \end{pmatrix} \end{array}$. For the current solution to remain optimal, all these entries must be ≥ 0: $45 - 2c_3 \ge 0$ so $c_3 \le 45/2 = 22.5$, $-30 + 2c_3 \ge 0$ so $c_3 \ge 15$, $90 - 4c_3 \ge 0$ so $c_3 \le 90/4 = 22.5$. Thus $15 \le c_3 \le 22.5$.

If $c_3 = 21$, that is in the interval where the current solution is optimal, so the optimal solution remains $s_1 = 24$, $x_3 = 8$, $x_1 = 2$, $x_2 = s_2 = s_3 = 0$, with $z = z^* = \mathbf{c}_{BV}^T \boldsymbol{\beta} = 288$.

On the other hand, $c_3 = 25$ is outside the interval. We have $\boldsymbol{\eta}_{NBV}^T = \begin{array}{c} \begin{array}{ccc} x_2 & s_2 & s_3 \end{array} \\ \begin{pmatrix} -5 & 20 & -10 \end{pmatrix} \end{array}$ and $z^* = 320$, so the tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	-5	0	0	20	-10	320	=	z
0	0	-2	0	1	2	-8	24	=	s_1
0	0	-2	1	0	2	-4	8	=	x_3
0	1	$5/4$	0	0	$-1/2$	$3/2$	2	=	x_1

Thus s_3, with the most negative entry in the z row, would enter the basis, and x_1 would leave. The next tableau would be

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	$20/3$	$10/3$	0	0	$50/3$	0	$1000/3$	=	z
0	$16/3$	$14/3$	0	1	$-2/3$	0	$104/3$	=	s_1
0	$8/3$	$4/3$	1	0	$2/3$	0	$40/3$	=	x_3
0	$2/3$	$5/6$	0	0	$-1/3$	1	$4/3$	=	s_3

which is optimal: $x_1 = x_2 = s_2 = 0$, $x_3 = 40/3$, $s_1 = 104/3$, $s_3 = 4/3$, $z = 1000/3$.

6.3.2. If $c_1 = 55$ (with $c_2 = 30$ and $c_3 = 20$ as in the original problem), we have $\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = \begin{pmatrix} 0 & 25/2 & 5/2 \end{pmatrix}$ and $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{array}{c} \begin{array}{ccc} x_2 & s_2 & s_3 \end{array} \\ \begin{pmatrix} -5/4 & 25/2 & 5/2 \end{pmatrix} \end{array}$. Thus we will want x_2 to enter the basis. Since $z^* = \mathbf{c}_{BV}^T \boldsymbol{\beta} = 270$, the tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	0	$-5/4$	0	0	$25/2$	$5/2$	270	=	z
0	0	-2	0	1	2	-8	24	=	s_1
0	0	-2	1	0	2	-4	8	=	x_3
0	1	$5/4$	0	0	$-1/2$	$3/2$	2	=	x_1

x_2 enters the basis and x_1 leaves the basis. The next tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	1	0	0	0	12	4	272	=	z
0	$8/5$	0	0	1	$6/5$	$-28/5$	$136/5$	=	s_1
0	$8/5$	0	1	0	$6/5$	$-8/5$	$56/5$	=	x_3
0	$4/5$	1	0	0	$-2/5$	$6/5$	$8/5$	=	x_2

which is optimal and has x_1 nonbasic, i.e. no desks are produced.

E.1. $B^{-1}N = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & s_3 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2 & -3/2 & -1/2 \\ 1 & 1/2 & -1/2 \end{pmatrix} \end{array}$ which goes in the main part of the tableau under x_1, x_2, s_3.

$\boldsymbol{\beta} = B^{-1}\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$, which goes in the main part of the rhs column.

$\mathbf{y}^T = \mathbf{c}_{BV}^T B^{-1} = (0 \quad 0 \quad 1)$, $\boldsymbol{\eta}_{NBV}^T = \mathbf{y}^T N - \mathbf{c}_{NBV}^T = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \end{array}$, and $z^* = \mathbf{c}_{BV}^T \boldsymbol{\beta} = 8$. Thus the tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs		
1	1	0	0	0	0	1	8	$=$	z
0	0	1/2	1	0	0	1/2	4	$=$	x_3
0	2	$-3/2$	0	1	0	$-1/2$	0	$=$	s_1
0	1	1/2	0	0	1	$-1/2$	2	$=$	s_2

E.2. The restriction of 20000 barrels of Crude 1 was the constraint I called `cr1bd`. In LINDO's solution, the "dual price" for this is given as 0.745904. This means that for small increases in the right side of the constraint, the profit increases by \$0.745904 per barrel, and thus the refinery should be willing to pay up to this much more than the regular price (\$24 per barrel), i.e. up to \$24.745904 per barrel, for small additional amounts of Crude 1.