Math 340: Solutions to Assignment 3

4.12.3. Note that we change the first constraint to $-x_1 - x_2 \leq -3$. We introduce slack variables s_1 and s_2 for the first two constraints, artificial variable a_3 for the third, and subtract the artificial variable a_0 from the first constraint where there is a negative rhs entry. The temporary objective w is $-a_0 - a_3$, and the initial tableau is

w	z	x_1	x_2	s_1	s_2	a_3	a_0	rhs		
1	0	0	0	0	0	1	1	0	=	w
0	1	−3	−1	0	0	0	0	0	=	z
0	0	−1	−1	1	0	0	−1	−3	=	s_1
0	0	2	1	0	1	0	0	4	=	s_2
0	0	1	1	0	0	1	0	3	=	a_3

Adjust the tableau so a_3 is basic, subtracting the a_3 row from the w row.

w	z	x_1	x_2	s_1	s_2	a_3	a_0	rhs		
1	0	−1	−1	0	0	0	1	−3	=	w
0	1	−3	−1	0	0	0	0	0	=	z
0	0	−1	−1	1	0	0	−1	−3	=	s_1
0	0	2	1	0	1	0	0	4	=	s_2
0	0	1	1	0	0	1	0	3	=	a_3

a_0 enters the basis and s_1 leaves, producing a tableau with a basic feasible solution (for the relaxed problem).

w	z	x_1	x_2	s_1	s_2	a_3	a_0	rhs		
1	0	−2	−2	1	0	0	0	−6	=	w
0	1	−3	−1	0	0	0	0	0	=	z
0	0	1	1	−1	0	0	1	3	=	a_0
0	0	2	1	0	1	0	0	4	=	s_2
0	0	1	1	0	0	1	0	3	=	a_3

Now ordinary pivots begin. x_1 enters (in a tie for most negative entry in the w row). s_2 leaves, w ith the smallest ratio.

w	z	x_1	x_2	s_1	s_2	a_3	a_0	rhs		
1	0	0	−1	1	1	0	0	−2	=	w
0	1	0	1/2	0	3/2	0	0	6	=	z
0	0	0	1/2	−1	−1/2	0	1	1	=	a_0
0	0	1	1/2	0	1/2	0	0	2	=	x_1
0	0	0	1/2	0	−1/2	1	0	1	=	a_3

Now x_2 enters. There's a tie for minimum ratio: a_0 leaves (in the lower-n umbered row).

w	z	x_1	x_2	s_1	s_2	a_3	a_0	rhs		
1	0	0	0	-1	0	0	2	0	=	w
0	1	0	0	1	2	0	-1	5	=	z
0	0	0	1	-2	-1	0	2	2	=	x_2
0	0	1	0	1	1	0	-1	1	=	x_1
0	0	0	0	1	0	1	-1	0	=	a_3

The basic solution is now feasible for the original problem, although a_3 is still basic (it has the value 0, as it should). We now remove the w row and column, and use z as the objective. We can also remove the a_0 column.

z	x_1	x_2	s_1	s_2	a_3	rhs		
1	0	0	1	2	0	5	=	z
0	0	1	-2	-1	0	2	=	x_2
0	1	0	1	1	0	1	=	x_1
0	0	0	1	0	1	0	=	a_3

Phase II can now begin, but it stops immediately: this tableau is optimal. The optimal solution is $x_1 = 1$, $x_2 = 2$, $s_1 = s_2 = a_3 = 0$, with $z = 5$.

4.12.6. This problem is a bit strange since the two constraints are equivalent and the objective is the left side of one of the constraints. Note that to minimize z I maximize $-z$, which I will call $\tilde{z}$. The slack variables for the constraints are both artificial. We need a Phase I, with temporary objective $w = -a_1 - a_2$. The initial tableau is

w	$\tilde{z}$	x_1	x_2	a_1	a_2	rhs		
1	0	0	0	1	1	0	=	w
0	1	1	1	0	0	0	=	$\tilde{z}$
0	0	1	1	1	0	2	=	a_1
0	0	2	2	0	1	4	=	a_2

To allow a_1 and a_2 to be basic, we must subtract their rows from the $\tilde{z}$ row:

w	$\tilde{z}$	x_1	x_2	a_1	a_2	rhs		
1	0	-3	-3	0	0	-6	=	w
0	1	1	1	0	0	0	=	$\tilde{z}$
0	0	1	1	1	0	2	=	a_1
0	0	2	2	0	1	4	=	a_2

x_1 enters the basis, and a_1 leaves (there are ties for both entering and leaving variable).

w	$\tilde{z}$	x_1	x_2	a_1	a_2	rhs		
1	0	0	0	3	0	0	=	w
0	1	0	0	-1	0	-2	=	$\tilde{z}$
0	0	1	1	1	0	2	=	x_1
0	0	0	0	-2	1	0	=	a_2

We have successfully completed Phase I, since $w = 0$ (even though the artificial variable a_2 is still basic, it has the value 0), so we can delete the w row and column. Since a_1 is artificial, it is not allowed to enter the basis even though its entry in the $\tilde{z}$ row is negative, and so this tableau is optimal. The optimal solution we have found is $x_1 = 2$, $x_2 = 0$, $a_1 = a_2 = 0$, $z = 2$ (it is not the only optimal solution: we could take $x_1 = 2 - x_2$ with $0 \le x_2 \le 2$).

4.14.2. The initial tableau is

z	x_1	x_2^U	s_1	s_2	rhs		
1	−2	−1	0	0	0	=	z
0	3	1	1	0	6	=	s_1
0	1	1	0	1	4	=	s_2

Since x_2 is URS, it has priority to enter the basis. It enters increasing, and s_2 leaves with the minimum ratio.

z	x_1	x_2^U	s_1	s_2	rhs		
1	−1	0	0	1	4	=	z
0	2	0	1	−1	2	=	s_1
0	1	1	0	1	4	=	x_2^U

Now x_1 enters and s_1 leaves. Note that no ratio needs to be calculated for x_2 because URS variables never leave the basis.

z	x_1	x_2^U	s_1	s_2	rhs		
1	0	0	1/2	1/2	5	=	z
0	1	0	1/2	−1/2	1	=	x_1
0	0	1	−1/2	3/2	3	=	x_2^U

This is optimal: $x_1 = 1$, $x_2 = 3$, $s_1 = s_2 = 0$, $z = 5$.

E.1. Multiply the third constraint by −1 to make the right side positive. We need the artificial variable a_0 in the first constraint. The temporary objective is $w = -a_2 - a_3 - a_0$. The tableau is

w	z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	0	0	0	0	0	0	1	1	1	0	=	w
0	1	7	−4	−10	−12	0	0	0	0	0	=	z
0	0	1	−3	0	−1	1	0	0	−1	−2	=	s_1
0	0	0	1	2	2	0	1	0	0	3	=	a_2
0	0	−1	1	0	1	0	0	1	0	1	=	a_3

To fix the a_3 column, we need to subtract the last two rows from the w row, so the w row becomes

w	z	x_1	x_2	x_3	x_4	s_1	s_2	a_3	a_0	rhs		
1	0	1	−2	−2	−3	0	0	0	1	−4	=	w

To make the basic solution feasible we need a special pivot where a_0 enters, and s_1 leaves.

w	z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	0	2	−5	−2	−4	1	0	0	0	−6	=	w
0	1	7	−4	−10	−12	0	0	0	0	0	=	z
0	0	−1	3	0	1	−1	0	0	1	2	=	a_0
0	0	0	1	2	2	0	1	0	0	3	=	a_2
0	0	−1	1	0	1	0	0	1	0	1	=	a_3

Now the simplex method can begin. x_2 enters and a_0 leaves, with the smallest ratio 2/3.

w	z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	0	1/3	0	−2	−7/3	−2/3	0	0	5/3	−8/3	=	w
0	1	17/3	0	−10	−32/3	−4/3	0	0	4/3	8/3	=	z
0	0	−1/3	1	0	1/3	−1/3	0	0	1/3	2/3	=	x_2
0	0	1/3	0	2	5/3	1/3	1	0	−1/3	7/3	=	a_2
0	0	−2/3	0	0	2/3	1/3	0	1	−1/3	1/3	=	a_3

Now x_4 enters and a_3 leaves.

w	z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	0	−2	0	−2	0	1/2	0	7/2	1/2	−3/2	=	w
0	1	−5	0	−10	0	4	0	16	−4	8	=	z
0	0	0	1	0	0	−1/2	0	−1/2	1/2	1/2	=	x_2
0	0	2	0	2	0	−1/2	1	−5/2	1/2	3/2	=	a_2
0	0	−1	0	0	1	1/2	0	3/2	−1/2	1/2	=	x_4

Now x_1 enters and a_2 leaves.

w	z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	0	0	0	0	0	0	1	1	1	0	=	w
0	1	0	0	−5	0	11/4	5/2	39/4	−11/4	47/4	=	z
0	0	0	1	0	0	−1/2	0	−1/2	1/2	1/2	=	x_2
0	0	1	0	1	0	−1/4	1/2	−5/4	1/4	3/4	=	x_1
0	0	0	0	1	1	1/4	1/2	1/4	−1/4	5/4	=	x_4

Since $w = 0$, this is feasible for the original problem. We discard the w row and column, and begin Phase II. x_3 enters and x_1 leaves.

z	x_1	x_2	x_3	x_4	s_1	a_2	a_3	a_0	rhs		
1	5	0	0	0	3/2	5	7/2	−3/2	31/2	=	z
0	0	1	0	0	−1/2	0	−1/2	1/2	1/2	=	x_2
0	1	0	1	0	−1/4	1/2	−5/4	1/4	3/4	=	x_3
0	−1	0	0	1	1/2	0	3/2	−1/2	1/2	=	x_4

This is now optimal (note that the artificial variable a_0 can't enter the basis). The optimal solution is $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/4$, $x_4 = 1/2$, $s_1 = a_2 = a_3 = 0$.

E.2. The initial tableau is

z	x_1	x_2	x_3^U	s_1	s_2	rhs		
1	−4	2	−3	0	0	0	=	z
0	2	1	1	1	0	1	=	s_1
0	1	−1	1	0	1	0	=	s_2

The basic solution is feasible, so we don't need Phase I. The URS variable x_3 has first priority to enter, and enters increasing; s_2 leaves with ratio 0.

z	x_1	x_2	x_3^U	s_1	s_2	rhs		
1	−1	−1	0	0	3	0	=	z
0	1	2	0	1	−1	1	=	s_1
0	1	−1	1	0	1	0	=	x_3^U

Now x_1 enters and s_1 leaves (there is no ratio to calculate for x_3, as we don't care if it becomes negative).

z	x_1	x_2	x_3^U	s_1	s_2	rhs		
1	0	1	0	1	2	1	=	z
0	1	2	0	1	-1	1	=	x_1
0	0	-3	1	-1	2	-1	=	x_3^U

This is optimal: $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $s_1 = s_2 = 0$, $z = 1$.

E.3. There are approximately $60 \times 60 \times 24 \times 365.25 = 3.156 \times 10^7$ seconds in a year, so the first computer can do about 3.156×10^{10} pivots in a year. The $n \times n$ Klee-Minty problem requires $2^n - 1$ pivots. Since $2^{34} - 1 < 3.156 \times 10^{10} < 2^{35} - 1$, the largest Klee-Minty problem this computer could do would be 34×34. The second computer could do about 3.156×10^{13} pivots in a year; it could do a 44×44 Klee-Minty problem. Typical $m \times n$ problems of this order of magnitude usually require something like $m \ln(n)$ pivots; $34 \ln(34) \approx 120$ pivots would take the first computer about 0.12 seconds, and $44 \ln(44) \approx 167$ pivots would take the second computer about 0.00017 seconds.