Math 340: Answers to Assignment 10

12.6.3. The total profit, i.e. revenue minus costs, is $f(q_1, q_2) = (200 - q_1 - q_2)(q_1 + q_2) - q_1 - .5q_2^2$. We want $0 = \dfrac{\partial f}{\partial q_1} = -2q_1 - 2q_2 + 199$ and $0 = \dfrac{\partial f}{\partial q_2} = -2q_1 - 3q_2 + 200$. Solving these equations, we get $q_1 = 98.5$, $q_2 = 1$. The Hessian matrix $\begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix}$ is negative definite, so f is concave and this is the global maximum.

12.6.6. With $f(x_1, x_2) = x_1^3 - 3x_1x_2^2 + x_2^4$ we have $0 = \dfrac{\partial f}{\partial x_1} = 3x_1^2 - 3x_2^2 = 3(x_1 - x_2)(x_1 + x_2)$ and $0 = \dfrac{\partial f}{\partial x_2} = -6x_1x_2 + 4x_2^3$. From the first equation, $x_2 = \pm x_1$. If $x_2 = x_1$, the second equation says $-6x_1^2 + 4x_1^3 = 0$ so $x_1 = 0$ or $3/2$. If $x_2 = -x_1$, the second equation says $6x_1^2 + 4x_1^3 = 0$, so again $x_1 = 0$ or $x_1 = -3/2$. Thus there are three critical points: $(0, 0)$, $(3/2, 3/2)$ and $(-3/2, 3/2)$. The Hessian matrix is $\begin{pmatrix} 6x_1 & -6x_2 \\ -6x_2 & -6x_1 + 12x_2^2 \end{pmatrix}$.

At $(0, 0)$ this matrix is all 0. But notice that $f(x_1, x_2) > 0$ for some (x_1, x_2) near $(0, 0)$ (e.g. $(t, 0)$) and < 0 for others (e.g. (t,t) where $t > 0$. So $(0, 0)$ is a saddle point.

At $(3/2, 3/2)$ the Hessian matrix is $\begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix}$ which is positive definite, so we have a local minimum.

At $(-3/2, 3/2)$ the Hessian matrix is $\begin{pmatrix} -9 & -9 \\ -9 & 36 \end{pmatrix}$ which is neither positive definite nor negative definite, and we have another saddle point.

12.9.1. The revenue is $(60 - 0.5p_1)p_1 + (40 - p_2)p_2$. If the capacity is c, then operating costs are $10c$, and we have constraints $c \geq 60 - 0.5p_1$ and $c \geq 40 - p_2$. We could also include constraints $p_1 \geq 0$, $p_2 \geq 0$, $p_1 \leq 120$ and $p_2 \leq 40$ (so the demand won't ever be negative), but I won't do so. The problem, in the form of (26), is

$$\begin{aligned} \text{maximize } & (60 - 0.5p_1)p_1 + (40 - p_2)p_2 - 10c \\ \text{subject to } & -c - .5p_1 \leq -60 \\ & -c - p_2 \leq -40 \end{aligned}$$

The KKT conditions are

$$\begin{aligned} 60 - p_1 + .5\lambda_1 &= 0 \\ 40 - 2p_2 + \lambda_2 &= 0 \\ -10 + \lambda_1 + \lambda_2 &= 0 \\ \lambda_1(-60 + c + .5p_1) &= 0 \\ \lambda_2(-40 + c + p_2) &= 0 \\ -c - .5p_1 &\leq -60 \\ -c - p_2 &\leq -40 \\ \text{all } \lambda_i &\geq 0 \end{aligned}$$

From the first three KKT equations, $p_1 = 60 + .5\lambda_1$, $\lambda_2 = 10 - \lambda_1$, and $p_2 = 20 - .5\lambda_2 = 15 + .5\lambda_1$.

If $\lambda_1 > 0$ we must have $-60 + c + .5p_1 = 0$ so $c = 60 - .5p_1 = 30 - .25\lambda_1$. In the fifth KKT condition we'd have $(10 - \lambda_1)(15 - .75\lambda_1) = 0$ so $\lambda_1 = 10$ or $\lambda_1 = 20$. But $\lambda_1 = 20$ would make $\lambda_2 = -10$ which is not allowed. For $\lambda_1 = 10$ we get a solution $p_1 = 65$, $p_2 = 20$, $c = 27.5$, $\lambda_2 = 0$.

On the other hand, if $\lambda_1 = 0$ we have $p_1 = 60$, $\lambda_2 = 10$, $p_2 = 25$, and $-15 + c = 0$ so $c_1 = 15$, but this violates the sixth KKT condition $-c - .5p_1 \leq -60$. So the only solution to the

KKT conditions is $p_1 = 65$, $p_2 = 20$, $c = 27.5$. Since the objective is concave and the constraints are linear, this solution must be the global maximum. The power company sets the prices at \$65 per kilowatt-hour at peak times and \$10 per kilowatt-hour at off-peak times, and maintains 27.5 kilowatt-hours of capacity. Of course, these numbers are wildly unrealistic.

12.9.10. The KKT conditions say

$$\begin{aligned} 2x_1 - 6 + \lambda_1 - \lambda_2 &= 0 \\ 2x_2 - 10 + \lambda_1 - \lambda_3 &= 0 \\ \lambda_1(7 - x_1 - x_2) &= 0 \\ \lambda_2 x_1 &= 0 \\ \lambda_3 x_2 &= 0 \\ x_1 + x_2 &\le 7 \\ -x_1 &\le 0 \\ -x_2 &\le 0 \\ \text{all } \lambda_i &\ge 0 \end{aligned}$$

Case 1: $\lambda_3 = 0$. The second condition says $\lambda_1 = 10 - 2x_2$.

Case 1(a): $\lambda_1 = 0$. Then $x_2 = 5$. The first condition says $\lambda_2 = 2x_1 - 6$ and then the fourth says $(2x_1 - 6)x_1 = 0$ so $x_1 = 0$ or $x_1 = 3$. But $x_1 = 0$ would make $\lambda_2 < 0$, while $x_1 = 3$ would violate the sixth condition.

Case 1(b): $\lambda_1 > 0$. Then $x_1 + x_2 = 7$, or $x_1 = 7 - x_2$. The first condition says $18 - 4x_2 - \lambda_2 = 0$ so $\lambda_2 = 18 - 4x_2$, and then the fourth says $(18 - 4x_2)(7 - x_2) = 0$, so either $x_2 = 4.5$ or $x_2 = 7$. $x_2 = 4.5$ produces a solution:

$x_1 = 2.5$, $x_2 = 4.5$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$

On the other hand, $x_2 = 7$ would make $\lambda_1 < 0$.

The objective is convex and the constraints are linear, so having one solution of the KKT conditions is enough: it is automatically the global minimum. But let's continue anyway.

Case 2: $\lambda_3 > 0$. Then $x_2 = 0$; the second KKT condition says $\lambda_1 = 10 + \lambda_3$. So $\lambda_1 > 0$, $x_1 = 7$, and $\lambda_2 = 0$. But then the first KKT condition says $8 + \lambda_1 = 0$, which is impossible.

E.1. The function $f(x, y)$ has gradient $\nabla f(x, y) = \begin{pmatrix} 2 - 2x + y - xy^2 \\ x - y - x^2 y \end{pmatrix}$ and Hessian $H(x, y) = \begin{pmatrix} -2 - y^2 & 1 - 2xy \\ 1 - 2xy & -1 - x^2 \end{pmatrix}$. Newton's method is the iteration $\mathbf{v}_{n+1} = \mathbf{v}_n - H(\mathbf{v}_n)^(- 1)\nabla f(\mathbf{v}_n)$. With $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have $\nabla f(\mathbf{v}_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $H(\mathbf{v}_0) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ so $\mathbf{v}_1 = \begin{pmatrix} 4/3 \\ 2/3 \end{pmatrix}$. Similarly, $\mathbf{v}_2 = \begin{pmatrix} 1.132401862 \\ .5362608116 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1.112294623 \\ .4980495355 \end{pmatrix}$. Since $\mathbf{v}_3$ is fairly close to $\mathbf{v}_2$, it does seem to be converging. The Hessian matrix at $\mathbf{v}_3$ is $H(\mathbf{v}_3) = \begin{pmatrix} -2.248053340 & -.107955641 \\ -.107955641 & -2.237199328 \end{pmatrix}$. Note that $-H(\mathbf{v}_3)$ is positive definite, so it appears to be converging to a local maximum.

E.2(a). The objective value at $(1, 0)$ is c_1, while the value at $(0, 0)$ is 0 and the value at $(0, 1)$ is c_2. So for $(1, 0)$ to be a global maximum, it's certainly necessary that $c_1 \ge c_2$ and $c_1 \ge 0$. On the other hand, if $c_1 \ge c_2$ and $c_1 \ge 0$ then

$$f(x_1, x_2) = c_1 x_1 + c_2 x_2^2 \le c_1 x_1 + c_1 x_2^2 \le c_1(x_1 + x_2) \le c_1$$

so this is also sufficient for $(1,0)$ to be a global maximum.
(b). Adding a slack variable s_1, the KKT conditions are

$-\lambda_1 + \lambda_2 = -c_1$
$2c_2x_2 - \lambda_1 + \lambda_3 = 0$
$x_1 + x_2 + s_1 = 1$
$\lambda_1 s_1 = \lambda_2 x_1 = \lambda_3 x_2 = 0$
$x_1, x_2, s_1, \lambda_1, \lambda_2, \lambda_3 \geq 0$

If $x_1 = 1$, $x_2 = 0$ is a solution, complementary slackness says $\lambda_2 = 0$, and so we must have $c_1 = \lambda_1 \geq 0$.

If $c_1 \geq 0$ and $c_1 \geq c_2$, we already know $(1,0)$ is a global maximum from (a), so it's also a local maximum.

If $c_2 > c_1 = 0$, $f(1,0) = 0$ but $f(x_1,x_2) > 0$ for points arbitrarily close to $(1,0)$ with $x_2 > 0$, so in this case $(1,0)$ is not a local maximum.

If $c_2 > c_1 > 0$ with $x_1 + x_2 \leq 1$, $f(x_1,x_2) \leq f(1-x_2,x_2) = c_1 - c_1x_2 + c_2x_2^2$, but $c_1x_2 > -c_2x_2^2$ if x_2 is small, so $f(x_1,x_2) < c_1 = f(1,0)$. Thus in this case $(1,0)$ is a local maximum.

So to sum up: $(1,0)$ is a local maximum if $c_2 \leq c_1 = 0$ or if $c_1 > 0$.

E.3. The KKT conditions are

$$\begin{aligned} L - 4\lambda_1 + \lambda_2 &= 0 \\ K - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1(8 - 4K - L) &= 0 \\ \lambda_2 K &= 0 \\ \lambda_3 L &= 0 \\ 4K + L &\leq 8 \\ K, L, \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

Case 1: If $\lambda_1 = 0$, the first KKT condition says $L + \lambda_2 = 0$, which implies $L = \lambda_2 = 0$, and the second says $K + \lambda_3 = 0$, which implies $K = \lambda_3 = 0$. The KKT conditions are indeed satisfied with $K = L = \lambda_1 = \lambda_2 = \lambda_3 = 0$, and the objective value at $K = L = 0$ is 0.

Case 2: If $\lambda_1 > 0$, $4K + L = 8$. Thus at least one of K and L is positive, implying that λ_2 or λ_3 is 0. If $\lambda_2 = 0$, $L = 4\lambda_1 > 0$, but that implies $\lambda_3 = 0$. Similarly, if $\lambda_3 = 0$, $K = \lambda_1 > 0$, but that implies $\lambda_2 = 0$. So we must have $\lambda_2 = \lambda_3 = 0$, $L = 4\lambda_1$ and $K = \lambda_1$. Then $4K + L = 8\lambda_1 = 8$, so $\lambda_1 = 1$, $K = 1$ and $L = 4$. The KKT conditions are satisfied with $K = 1$, $L = 4$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, and the objective value is 4. This agrees with LINGO's solution shown in Figure 2 on page 661.

E.4(a). We are maximizing a non-constant linear (and thus concave) function subject to constraints $g_i(x_1,x_2) \leq b_i$ where g_i are convex functions). Any local maximum is thus a global maximum. If there were two local maxima, all points of the straight line segment joining them would have to be maxima as well, and couldn't lie in the interior of the feasible region (because the gradient is not 0). So this line segment would have to be in the boundary of the feasible region. But the boundary of the feasible region is composed of pieces of the curves $x_1^2 + 3x_2^2 = 4$ and $x_1^2 - x_2 = 0$, and has no straight segments.
(b). The KKT conditions are

$2 - 2x_1\lambda_1 - 2x_1\lambda_2 = 0$

$3 - 6x_2\lambda_1 + \lambda_2 = 0$
$x_1^2 + 3x_2^2 \le 4$
$x_1^2 - x_2 \le 0$
$\lambda_1(4 - x_1^2 - 3x_2^2) = 0$
$\lambda_2(x_1^2 - x_2) = 0$
$\lambda_1, \lambda_2 \ge 0$

From the second condition, $6x_2\lambda_1 = 3 + \lambda_2 > 0$ so $\lambda_1 > 0 and x_2 > 0$. Therefore $x_1^2 + 3x_2^2 = 4$.

Case 1: $\lambda_2 = 0$. Then the first KKT condition says $x_1\lambda_1 = 1$ and the second says $x_2\lambda_1 = 1/2$, so $x_1 = 2x_2$. So $x_1^2 + 3x_2^2 = 7x_2^2 = 4$, i.e. $x_2 = 2/\sqrt{7}$, $x_1 = 4/\sqrt{7}$, $\lambda_1 = 1/x_1 = \sqrt{7}/4$. But then $x_1^2 - x_2 = 16/7 - 2/\sqrt{7} > 0$, so this is not a solution.

Case 2: $x_1^2 - x_2 = 0$. Now with $x_2 = x_1^2$ we have $x_1^2 + 3x_2^2 = x_2 + 3x_2^2 = 4$. This has roots $x_2 = 1$ and $x_2 = -4/3$. But since we know $x_2 > 0$ it must be $x_2 = 1$ and $x_1 = \pm 1$. The first KKT condition says $x_1(\lambda_1 + \lambda_2) = 1$, so it must be $x_1 = 1$, and $\lambda_1 + \lambda_2 = 1$. From the second KKT condition, $6\lambda_1 - \lambda_2 = 3$. Solving these two linear equations for λ_1 and $\lambda 2$, we get $\lambda_1 = 4/7$, $\lambda_2 = 3/7$, which is feasible. Thus we have a solution of the KKT equations:

$$x_1 = x_2 = 1,\ \lambda_1 = 4/7,\ \lambda_2 = 3/7$$

(c). The Lingo file should look like this:

```
model:
max=2*x1+3*x2;
x1^2 + 3*x2^2 <= 4;
x1^2 - x2 <= 0;
@free(x1);
@free(x2);
end
```

Lingo's output is

```
Local optimal solution found at step:             10
Objective value:                                5.000000

       Variable           Value        Reduced Cost
         X1        0.9999999            0.0000000
         X2         1.000000            0.0000000

       Row    Slack or Surplus      Dual Price
        1         5.000000            1.000000
        2        0.2279561E-06       -0.5714284
        3        0.3080988E-06       -0.4285715
```

This agrees (with an allowance for roundoff error) with our $x_1 = x_2 = 1$; the dual prices for rows 2 and 3 correspond to our $\lambda_1 = 4/7$ and $\lambda_2 = 3/7$ (with a different sign convention).