The Dual Simplex Method

We are only considering Phase II of the Dual Simplex Method. So the assumption is that we begin with a tableau (possibly the initial tableau of the problem, possibly something that arises during sensitivity analysis as a result of some change in the problem) that is dual-feasible but not primal-feasible, i.e. the basic solution of the dual problem is feasible, but the basic solution of the primal is not. Thus the entries for non-artificial variables in the z row are all ≥ 0, and any entries for URS variables there are 0. This fact will continue to be true in all subsequent pivots. If we get to a tableau where the basic solution of the primal problem is feasible, it will be optimal. I will also assume (for the sake of simplicity) that all URS variables are basic and all artificial variables are nonbasic. Then the only consideration we have to give to these is that URS variables are ignored when we look for candidates to leave the basis, and artificial variables are ignored when we look for candidates to enter the basis. Here, then, is the procedure for the Dual Simplex Method:

1. Find a basic variable (not URS) whose value in the basic solution is negative. This will be the leaving variable. Usually we take the one with the most negative value (corresponding to the most-negative-entry rule). If there is no leaving variable, stop: the current tableau is optimal.
2. If x_L is the leaving variable, calculate the ratios (entry in z row)/($-$entry in x_L row) for all non-artificial variables whose entries in the x_L row is negative. The entering variable has the minimum ratio. If there are no ratios to calculate, stop: the problem is infeasible.
3. Pivot in the usual way, with the entering variable entering and the leaving variable leaving the basis, and return to step (1).

Worked Example:

$$
\begin{array}{llrrrrl}
\text{maximize} & z = & -x_1 & -2x_2 & -x_3 & & \\
\text{subject to} & & 3x_1 & -x_2 & -x_3 & & \leq -3 \\
 & & x_1 & & & -4x_4 & \leq -2 \\
 & & -3x_1 & +2x_2 & +x_3 & +2x_4 & \leq 6
\end{array}
$$

all variables ≥ 0

Initial tableau:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	1	2	1	0	0	0	0	0	$=$	z
0	3	-1	-1	0	1	0	0	-3	$=$	s_1
0	1	0	0	-4	0	1	0	-2	$=$	s_2
0	-3	2	1	2	0	0	1	6	$=$	s_3

s_1 leaves; ratios $2/1$ for x_2, $1/1$ for x_3. With minimum ratio, x_3 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	4	1	0	0	1	0	0	-3	$=$	z
0	-3	1	1	0	-1	0	0	3	$=$	x_3
0	1	0	0	-4	0	1	0	-2	$=$	s_2
0	0	1	0	2	1	0	1	3	$=$	s_3

s_2 leaves; ratio of 0 for x_4, so x_4 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	4	1	0	0	1	0	0	-3	$=$	z
0	-3	1	1	0	-1	0	0	3	$=$	x_3
0	$-1/4$	0	0	1	0	$-1/4$	0	$1/2$	$=$	x_4
0	$1/2$	1	0	0	1	$1/2$	1	2	$=$	s_3

Optimal solution: $x_1 = x_2 = 0$, $x_3 = 3$, $x_4 = 1/2$, $s_1 = s_2 = 0$, $s_3 = 2$, $z = -3$.