Complementary Slackness

Consider the problem

maximize

$$2x_1 + 16x_2 + 2x_3$$

subject to

$$\begin{array}{rrrl} 2x_1 & +x_2 & -x_3 & \leq -3 \\ -3x_1 & +x_2 & +2x_3 & \leq 12 \\ & & x_1, x_2, x_3 & \geq 0 \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

#1: $x_1 = 6$, $x_2 = 0$, $x_3 = 12$

#2: $x_1 = 0$, $x_2 = 2$, $x_3 = 5$

#3: $x_1 = 0$, $x_2 = 0$, $x_3 = 6$

For what, if any, values of c_2 and c_3 would $x_1 = 0$, $x_2 = 0$, $x_3 = 6$ be an optimal solution?

Solution:

The equations of the dual problem are

$$\begin{aligned} 2y_1 - 3y_2 - \eta_1 &= 2 \\ y_1 + y_2 - \eta_2 &= 16 \\ -y_1 + 2y_2 - \eta_3 &= 2 \end{aligned}$$

Given values of x_1, x_2, x_3, we first calculate the values of the slack variables s_1 and s_2. Of course, the solution must be feasible to be optimal: in this case that means the decision and slack variables all ≥ 0. Next, if it is optimal, whenever an x_j or s_i is nonzero, complementary slackness says the corresponding η_j or y_i must be 0. We see what's left of the equations of the dual when those variables are set to 0. If we can find a solution that is feasible (again all $y_i \geq 0$ and $\eta_j \geq 0$), then we conclude the alleged solution is optimal. If there is no feasible solution, it isn't.

#1: With $x_1 = 6$, $x_2 = 0$ and $x_3 = 12$, $s_1 = -3$. This is not feasible, therefore not optimal.

#2: With $x_1 = 0$, $x_2 = 2$, $x_3 = 5$, $s_1 = 0$ and $s_2 = 0$: the solution of the primal is feasible. From complementary slackness, $\eta_2 = 0$ and $\eta_3 = 0$. The equations of the dual become

$$\begin{aligned} 2y_1 - 3y_2 - \eta_1 &= 2 \\ y_1 + y_2 \qquad &= 16 \\ -y_1 + 2y_2 \qquad &= 2 \end{aligned}$$

From the second and third, $y_1 = 10$, $y_2 = 6$, and then from the first, $\eta_1 = 0$. All ≥ 0, so this solution of the dual is feasible. Conclusion: yes, solution #2 is optimal.

#3: With $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $s_1 = 3$ and $s_2 = 0$: the solution of the primal is feasible. From complementary slackness, $\eta_3 = 0$ and $y_1 = 0$. The equations of the dual become

$$\begin{aligned} -3y_2 - \eta_1 &= 2 \\ +y_2 - \eta_2 &= 16 \\ +2y_2 \qquad &= 2 \end{aligned}$$

The third equation says $y_2 = 1$, then from the first equation $\eta_1 = -5$. The solution of the dual is not feasible. Conclusion: no, solution #3 is not optimal.

In the last part, we again take $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, so $s_1 = 3$ and $s_2 = 0$, but c_2 and c_3 are unspecified (c_1 is kept at 2). With complementary slackness telling us $\eta_3 = 0$ and $y_1 = 0$, the equations of the dual become

$$\begin{aligned} -3y_2 - \eta_1 &= 2 \\ +y_2 - \eta_2 &= c_2 \\ +2y_2 \qquad &= c_3 \end{aligned}$$

But if $y_2 \geq 0$ and $\eta_2 \geq 0$, the left side of the first equation ≤ 0. So this is impossible: there are no such values of c_2 and c_3.