Non-homogeneous Sturm-Liouville problems

Non-homogeneous Sturm-Liouville problems can arise when trying to solve non-homogeneous PDE’s. For example, consider a radially-symmetric non-homogeneous heat equation in polar coordinates:

\[ u_t = u_{rr} + \frac{1}{r}u_r + h(r)e^{-\mu t} \]

with boundary conditions \( c_1 u(a, t) + c_2 u_r(a, t) = 0 \) and \( d_1 u(b, t) + d_2 u_r(b, t) = 0 \). If we look for a solution of the form \( u(r, t) = y(r)e^{-\mu t} \), we get

\[ -\mu y' = y'' + \frac{1}{r}y' + h \]

or the non-homogeneous Sturm-Liouville problem

\[ (ry')' + \mu ry = -rh \]

with boundary conditions \( c_1 y(a) + c_2 y'(a) = 0 \) and \( d_1 y(b) + d_2 y'(b) = 0 \).

One way to solve such problems is using the eigenfunction expansion for the corresponding homogeneous Sturm-Liouville problem. Consider the non-homogeneous Sturm-Liouville problem

\[ (p(x)y')' + q(x)y + \mu r(x)y = h(x) \]

with boundary conditions \( c_1 y(a) + c_2 y'(a) = 0 \) and \( d_1 y(b) + d_2 y'(b) = 0 \).

Suppose \( \lambda_n \) and \( y_n \) are the eigenvalues and eigenfunctions of the homogeneous problem

\[ (p(x)y')' + q(x)y + \lambda r(x)y = 0 \]

with the same boundary conditions.

Suppose we can write \( y(x) = \sum_n b_n y_n(x) \). Since \( (p(x)y_n')' + q(x)y_n + \lambda_n r(x)y_n = 0 \) we have

\[ (p(x)y_n')' + q(x)y_n + \mu r(x)y_n = (\mu - \lambda_n)r(x)y_n \]

Thus we get

\[ (p(x)y)' + q(x)y + \mu r(x)y = \sum_n b_n ((p(x)y_n')' + q(x)y_n + \mu r(x)y_n) \]

\[ h(x) = \sum_n b_n (\mu - \lambda_n)r(x)y_n \]

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i.e. if \( h(x)/r(x) = \sum c_n y_n \), we have \( b_n(\mu - \lambda_n) = c_n \). The coefficients \( c_n \) in the eigenfunction expansion of \( h(x)/r(x) \) are obtained as usual by

\[
c_n = \frac{\int_a^b (h(x)/r(x)) y_n(x) r(x) \, dx}{\int_a^b y_n(x)^2 r(x) \, dx} = \frac{\int_a^b h(x) y_n(x) \, dx}{\int_a^b y_n(x)^2 r(x) \, dx}
\]

For example, consider the nonhomogeneous Sturm-Liouville problem

\[
y'' + 2y = x, \quad y(0) = 0, y(1) = 0
\]

The corresponding homogeneous problem

\[
y'' + \lambda y = 0, \quad y(0) = 0, y(1) = 0
\]

has eigenvalues \( \lambda_n = (n\pi)^2 \) and eigenfunctions \( y_n = \sin(n\pi x) \). The eigenfunction expansion is the Fourier sine series on the interval \([0, 1]\). We get

\[
c_n = 2 \int_0^1 x \sin(n\pi x) \, dx = \frac{2}{n\pi} (-1)^{n+1}
\]

and so

\[
b_n = \frac{2}{n\pi(2 - n^2\pi^2)} (-1)^{n+1}
\]

\[
y = \sum_{n=1}^{\infty} b_n \sin(n\pi x)
\]

Of course, if \( \mu \) happens to be an eigenvalue, say \( \lambda_k \), of the homogeneous Sturm-Liouville problem, the formula \( b_k(\mu - \lambda_k) = c_k \) can’t be solved for \( b_k \). In fact, the non-homogeneous problem has no solution if \( \mu = \lambda_k \) and \( c_k \neq 0 \). On the other hand, if \( \mu = \lambda_k \) and \( c_k = 0 \), then \( b_k \) is arbitrary (you can always add a solution of the homogeneous problem to a solution of the non-homogeneous problem and get another solution).

For example, consider

\[
y'' + 4\pi^2 y = x, \quad y(0) = 0, y(1) = 0
\]

Since \( \mu = 4\pi^2 = \lambda_2 \), and \( c_2 = \frac{1}{\pi} \neq 0 \), there is no solution. In fact, let’s see this directly from the differential equation. The general solution is \( y = x/(4\pi^2) + A \cos(2\pi x) + B \sin(2\pi x) \); from the boundary condition at \( x = 0 \) we get \( A = 0 \); and then at \( x = 1 \) we have \( y(1) = 1/(4\pi^2) \neq 0 \).

On the other hand, consider

\[
y'' + 4\pi^2 y = 1, \quad y(0) = 0, y(1) = 0
\]
In this case \( c_2 = 2 \int_0^1 \sin(2\pi x) \, dx = 0 \), so there should be solutions where the coefficient of \( \sin(2\pi x) \) is arbitrary: the general solution of \( y'' + 4\pi^2 y = 1 \) is \( y = \frac{1}{4\pi^2} + A \cos(2\pi x) + B \sin(2\pi x) \), from \( x = 0 \) we get \( A = -1/(4\pi^2) \), and then the boundary condition at \( x = 1 \) is also satisfied no matter what \( B \) is.