

Lesson 34: Padé meets Chebyshev

```
> restart;
with(numtheory):
```

Continued fractions for numbers

Consider any positive number x . We can represent it by a simple continued fraction with integer elements as follows. Let $b_0 = \text{floor}(x)$, so $0 \leq x - b_0 < 1$. If that is 0, then $x = b_0$. Otherwise

$x = b_0 + \frac{1}{x_1}$ with $x_1 > 1$. Let $b_1 = \text{floor}(x_1) \geq 1$. Again, if $b_1 = x_1$, then $x = b_0 + \frac{1}{b_1}$, otherwise

$x = b_0 + \frac{1}{b_1 + \frac{1}{x_2}}$ etc.

The continued fraction for x terminates if and only if x is a rational number. For example:

```
> cfrac([3,7,110,3]); CFrac([3,7,110,3]);
```

$$3 + \frac{\frac{7291}{2320}}{7 + \frac{1}{110 + \frac{1}{3}}} \quad (1.1)$$

For an irrational number, the continued fraction does not terminate. It is eventually periodic if and only if x is a quadratic irrational, i.e. an irrational root of a quadratic polynomial with integer coefficients.

```
> cfrac(sqrt(3));
```

$$1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}}}}}}}}}} \quad (1.2)$$

```
> cfrac(sqrt(7)/5);
```

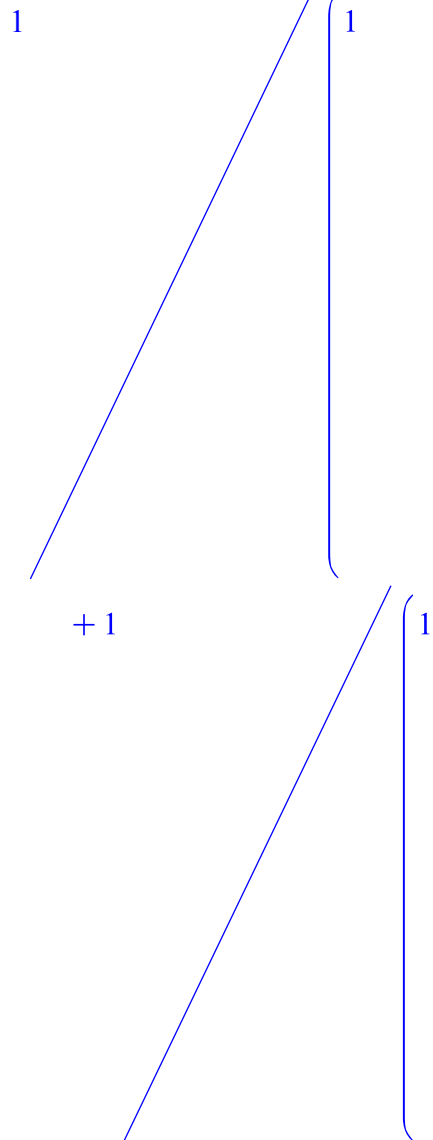
(1.3)

$$\begin{aligned}
 & \frac{1}{1 + \frac{1}{1 + \frac{1}{8 + \frac{1}{13 + \frac{1}{8 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{8 + \frac{1}{13 + \dots}}}}}}}}}}}}
 \end{aligned}$$

(1.3)

You can get as many terms as you want.

> cfrac(sqrt(7)/5,15);



(1.4)

- $\frac{a}{b}$ in lowest terms with $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$ is a convergent of x .

So e.g. for π we get the following good approximations:

```
> S:= [seq(nthconver(cfrac(Pi,5),n),n=1..5)];
```

$$S := \left[\frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215} \right] \quad (1.12)$$

```
> Digits:= 15; seq(evalf(abs(Pi - S[n])),n=1..5);
```

Digits := 15

$$0.00126448926735, 0.00008321962753, 2.6676419 \cdot 10^{-7}, 5.7789 \cdot 10^{-10}, 3.3163 \cdot 10^{-10} \quad (1.13)$$

```
> seq(evalf(abs(Pi-S[n])*denom(S[n])^2), n=1..5);
```

$$0.061959974100, 0.9350557349, 0.0034063120, 0.63322, 0.36587 \quad (1.14)$$

Continued fractions for analytic functions

Consider again our function f_0 .

```
> f0:= x -> arctan(2*x+1);
```

$$f_0 := x \rightarrow \arctan(2x + 1) \quad (2.1)$$

```
> Q:=cfrac(f0(x),x,21,quotients);
```

$$Q := \left[\frac{1}{4} \pi, [x, 1], [x, 1], [-x, 3], [4x, 1], [-x, 5], [6x, 1], [-3x, 14], [17x, 1], [-32x, \right. \quad (2.2)$$

153], [185x, 1], [-85x, 407], [492x, 1], [-111x, 533], [644x, 1], [-287x, 1380],
[1667x, 1], [-5888x, 28339], [34227x, 1], [-15003x, 72257], [87260x, 1], [
-19015x, 91623], ...]

```
> CFRAC(Q[1..5]);
```

$$\frac{1}{4} \pi + \frac{x}{1 + \frac{x}{1 - \frac{x}{3 + \frac{4x}{1}}}} \quad (2.3)$$

How do these come about? In a way it's rather analogous to what we had for numbers.

```
> f0(0);
```

$$\frac{1}{4} \pi \quad (2.4)$$

```
> f1:= solve(f0(x) = Pi/4 + x/f1,f1); series(f1, x);
```

$$f_1 := \frac{4x}{4 \arctan(2x + 1) - \pi}$$

$$1 + x + \frac{1}{3} x^2 - \frac{1}{3} x^3 + \frac{11}{45} x^4 + O(x^5) \quad (2.5)$$

```
> f2:= solve(f1 = 1 + x/f2, f2); series(f2,x);
```

$$f2 := -\frac{x(4 \arctan(2x+1) - \pi)}{-4x + 4 \arctan(2x+1) - \pi}$$

$$1 - \frac{1}{3}x + \frac{4}{9}x^2 - \frac{68}{135}x^3 + O(x^4) \quad (2.6)$$

> `f3:= solve(f2 = 1 + x/f3, f3); series(f3,x);`

$$f3 := -\frac{x(-4x + 4 \arctan(2x+1) - \pi)}{4x \arctan(2x+1) - x\pi - 4x + 4 \arctan(2x+1) - \pi}$$

$$-3 - 4x - \frac{4}{5}x^2 + O(x^3) \quad (2.7)$$

> `f4:= solve(f3 = -3 + x/f4, f4); series(f4,x);`

$$f4 := \frac{x(4x \arctan(2x+1) - x\pi - 4x + 4 \arctan(2x+1) - \pi)}{4x^2 + 8x \arctan(2x+1) - 2x\pi - 12x + 12 \arctan(2x+1) - 3\pi}$$

$$-\frac{1}{4} + \frac{1}{20}x + O(x^2) \quad (2.8)$$

And thus

$$f0(x) = \frac{\pi}{4} + \frac{x}{f1} = \frac{\pi}{4} + \frac{x}{1 + \frac{x}{f2}} = \frac{\pi}{4} + \frac{x}{1 + \frac{x}{1 + \frac{x}{f3}}} = \frac{\pi}{4} + \frac{x}{1 + \frac{x}{-3 + \frac{x}{f4}}}$$

$$= \frac{\pi}{4} + \frac{x}{1 + \frac{x}{-3 + \frac{x}{-\frac{1}{4} + \frac{x}{\dots}}}}$$

which Maple prefers to write as

> `CFRAC([(1/4)*Pi, [x, 1], [x, 1], [-x, 3], [4*x, 1], `...`]);`

$$\frac{1}{4} \pi + \frac{x}{1 + \frac{x}{1 - \frac{x}{3 + \frac{4x}{1 + \dots}}}} \quad (2.9)$$

> `for nn from 1 to 20 do F[nn]:= nthconver(Q,nn) end do;`

$$F_1 := x + \frac{1}{4} \pi$$

$$F_2 := \frac{\frac{1}{4} \pi x + x + \frac{1}{4} \pi}{x + 1}$$

$$F_3 := \frac{-x^2 + \frac{1}{2} \pi x + 3x + \frac{3}{4} \pi}{2x + 3}$$

$$F_4 := \frac{\pi x^2 + 3x^2 + \frac{3}{2}\pi x + 3x + \frac{3}{4}\pi}{4x^2 + 6x + 3}$$

$$F_5 := \frac{x^3 + \frac{9}{2}\pi x^2 + 12x^2 + \frac{27}{4}\pi x + 15x + \frac{15}{4}\pi}{18x^2 + 27x + 15}$$

$$F_6 := \frac{6\pi x^3 + 19x^3 + \frac{27}{2}\pi x^2 + 30x^2 + \frac{45}{4}\pi x + 15x + \frac{15}{4}\pi}{24x^3 + 54x^2 + 45x + 15}$$

$$F_7 := \frac{-3x^4 + \frac{141}{2}\pi x^3 + 230x^3 + \frac{675}{4}\pi x^2 + 375x^2 + \frac{585}{4}\pi x + 210x + \frac{105}{2}\pi}{282x^3 + 675x^2 + 585x + 210}$$

$$F_8 :=$$

$$\frac{1}{408x^4 + 1200x^3 + 1440x^2 + 840x + 210} \left(320x^4 + 102\pi x^4 + 300\pi x^3 + 740x^3 + 360\pi x^2 + 630x^2 + 210x + 210\pi x + \frac{105}{2}\pi \right)$$

$$F_9 := \left(96x^5 + 41600x^4 + 13350\pi x^4 + 101220x^3 + 40500\pi x^3 + 89670x^2 + 50400\pi x^2 + 30450\pi x + 32130x + \frac{16065}{2}\pi \right) / (53400x^4 + 162000x^3 + 201600x^2 + 121800x + 32130)$$

$$F_{10} := \left(59296x^5 + 18870\pi x^5 + 178500x^4 + 68850\pi x^4 + 217770x^3 + 107100\pi x^3 + 128520x^2 + 89250\pi x^2 + \frac{80325}{2}\pi x + 32130x + \frac{16065}{2}\pi \right) / (75480x^5 + 275400x^4 + 428400x^3 + 357000x^2 + 160650x + 32130)$$

$$F_{11} := \left(-8160x^6 + 20597472x^5 + 6545340\pi x^5 + 64045800x^4 + 24579450\pi x^4 + 39305700\pi x^3 + 81010440x^3 + 49576590x^2 + 33736500\pi x^2 + 13076910x + 15663375\pi x + \frac{6538455}{2}\pi \right) / (26181360x^5 + 98317800x^4 + 157222800x^3 + 134946000x^2 + 62653500x + 13076910)$$

$$F_{12} := \left(9284040\pi x^6 + 29165472x^6 + 40419540\pi x^5 + 108419472x^5 + 171188640x^4 + 77272650\pi x^4 + 144242280x^3 + 83216700\pi x^3 + 65384550x^2 + 53496450\pi x^2 + 19615365\pi x + 13076910x + \frac{6538455}{2}\pi \right) / (37136160x^6 + 161678160x^5 + 309090600x^4 + 332866800x^3 + 213985800x^2 + 78461460x + 13076910)$$

$$\begin{aligned}
F_{13} := & \left(905760 x^7 + 4221860580 \pi x^6 + 13258877184 x^6 + 50678494776 x^5 \right. \\
& + 18815295870 \pi x^5 + 82251386280 x^4 + 36823389750 \pi x^4 + 40609749600 \pi x^3 \\
& + 71378133750 x^3 + 33398428140 x^2 + 26774973225 \pi x^2 + 6969993030 x \\
& \left. + \frac{20184210585}{2} \pi x + \frac{3484996515}{2} \pi \right) / \left(16887442320 x^6 + 75261183480 x^5 \right. \\
& + 147293559000 x^4 + 162438998400 x^3 + 107099892900 x^2 + 40368421170 x \\
& \left. + 6969993030 \right)
\end{aligned}$$

$$\begin{aligned}
F_{14} := & \left(5978921760 \pi x^7 + 18783469728 x^7 + 30252044340 \pi x^6 + 83081017152 x^6 \right. \\
& + 160923978936 x^5 + 68578882470 \pi x^5 + 175143414600 x^4 + 90414944550 \pi x^4 \\
& + 113485783950 x^3 + 75061463400 \pi x^3 + 41819958180 x^2 + 39407268285 \pi x^2 \\
& \left. + \frac{24394975605}{2} \pi x + 6969993030 x + \frac{3484996515}{2} \pi \right) / \left(23915687040 x^7 \right. \\
& + 121008177360 x^6 + 274315529880 x^5 + 361659778200 x^4 + 300245853600 x^3 \\
& \left. + 157629073140 x^2 + 48789951210 x + 6969993030 \right)
\end{aligned}$$

$$\begin{aligned}
F_{15} := & \left(-259953120 x^8 + 7039238042340 \pi x^7 + 22115890472832 x^7 \right. \\
& + 36347831274510 \pi x^6 + 100107075669048 x^6 + 198468943069320 x^5 \\
& + 84070544950350 \pi x^5 + 221212387761750 x^4 + 113117625343800 \pi x^4 \\
& + 147025032974820 x^3 + 95900402176425 \pi x^3 + 55711154288790 x^2 \\
& \left. + \frac{102971192028705}{2} \pi x^2 + 9618590381400 x + \frac{32664872335095}{2} \pi x \right. \\
& \left. + 2404647595350 \pi \right) / \left(28156952169360 x^7 + 145391325098040 x^6 \right. \\
& + 336282179801400 x^5 + 452470501375200 x^4 + 383601608705700 x^3 \\
& \left. + 205942384057410 x^2 + 65329744670190 x + 9618590381400 \right)
\end{aligned}$$

$$\begin{aligned}
F_{16} := & \left(31311784083456 x^8 + 9966862573920 \pi x^8 + 57469395957120 \pi x^7 \right. \\
& + 160611946065216 x^7 + 150668828352000 \pi x^6 + 368367348555360 x^6 \\
& + 490433015207520 x^5 + 234792257515200 \pi x^5 + 410393189606400 x^4 \\
& + 238245084831600 \pi x^4 + 161592318407520 \pi x^3 + 216738903260880 x^3 \\
& + 67330132669800 x^2 + 71818808181120 \pi x^2 + 9618590381400 x \\
& \left. + 19237180762800 \pi x + 2404647595350 \pi \right) / \left(39867450295680 x^8 \right.
\end{aligned}$$

$$\begin{aligned}
& + 229877583828480 x^7 + 602675313408000 x^6 + 939169030060800 x^5 \\
& + 952980339326400 x^4 + 646369273630080 x^3 + 287275232724480 x^2 \\
& + 76948723051200 x + 9618590381400)
\end{aligned}$$

$$\begin{aligned}
F_{17} := & (1530603970560 x^9 + 241003884889020960 \pi x^8 + 757126286037024768 x^8 \\
& + 1414609181484508800 \pi x^7 + 3962151478002801600 x^7 \\
& + 3774796557999667200 \pi x^6 + 9270577153918190880 x^6 \\
& + 5987741207698958400 \pi x^5 + 12595882678824725280 x^5 \\
& + 10764449206100029440 x^4 + 6186965891027922000 \pi x^4 \\
& + 4276217522018201760 \pi x^3 + 5814136503057682800 x^3 \\
& + 1939107820890240000 \pi x^2 + 1851434369563779000 x^2 \\
& + 531003900595568400 \pi x + 272581232818494600 x + 68145308204623650 \pi) / \\
& (964015539556083840 x^8 + 5658436725938035200 x^7 + 15099186231998668800 x^6 \\
& + 23950964830795833600 x^5 + 24747863564111688000 x^4 \\
& + 17104870088072807040 x^3 + 7756431283560960000 x^2 + 2124015602382273600 x \\
& + 272581232818494600)
\end{aligned}$$

$$\begin{aligned}
F_{18} := & (1071709964428419072 x^9 + 341135805317559840 \pi x^9 + 6254391364011172800 x^8 \\
& + 2208008900313367200 \pi x^8 + 6571551169488412800 \pi x^7 \\
& + 16570260717007108320 x^7 + 11811031155972417600 \pi x^6 \\
& + 26056627965425977920 x^6 + 14142155726230131600 \pi x^5 \\
& + 26642410379482978080 x^5 + 18182771648010169200 x^4 \\
& + 11717786173162109040 \pi x^4 + 8118644953946927400 x^3 \\
& + 6734359869633396000 \pi x^3 + 2180649862547956800 x^2 \\
& + 2597538806858595600 \pi x^2 + 613307773841612850 \pi x + 272581232818494600 x \\
& + 68145308204623650 \pi) / (1364543221270239360 x^9 + 8832035601253468800 x^8 \\
& + 26286204677953651200 x^7 + 47244124623889670400 x^6 \\
& + 56568622904920526400 x^5 + 46871144692648436160 x^4 \\
& + 26937439478533584000 x^3 + 10390155227434382400 x^2 \\
& + 2453231095366451400 x + 272581232818494600)
\end{aligned}$$

$$\begin{aligned}
F_{19} := & (-22963651370311680 x^{10} + 66079381230290794291200 x^9 \\
& + 21033668599840939896000 \pi x^9 + 138320717560130888244000 \pi x^8 \\
& + 392479398164879280604800 x^8 + 1058230859588548008105600 x^7 \\
& + 418207300094055236688000 \pi x^7 + 763595596897991505648000 \pi x^6 \\
& + 1693797739067377533189600 x^6 + 929046697047118705255200 \pi x^5 \\
& + 1763601615351182805438240 x^5 + 782535984031335431898000 \pi x^4)
\end{aligned}$$

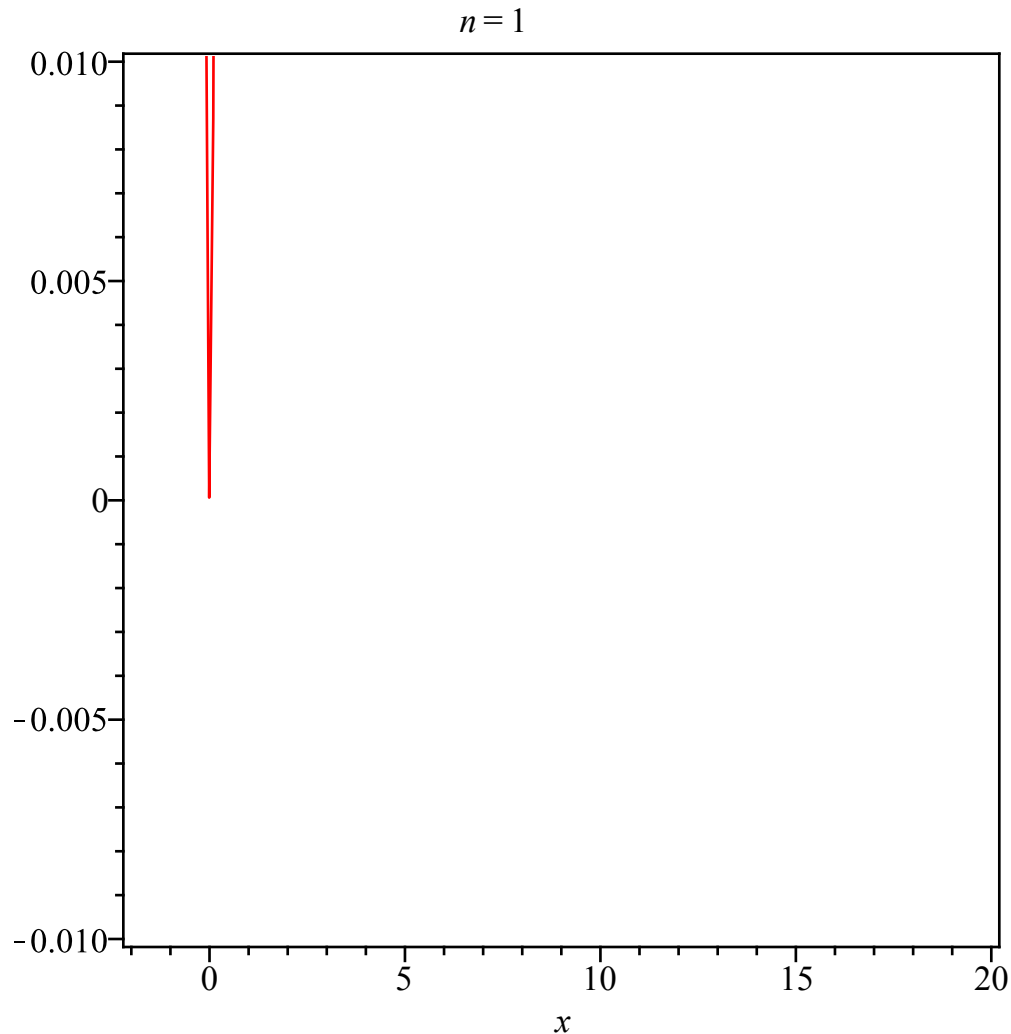
$$\begin{aligned}
& + 1226603041014896380836000 x^4 + 457512206463284024052000 \pi x^3 \\
& + 558851858590777756804800 x^3 + 179723710046546229564000 \pi x^2 \\
& + 153477680882151840013800 x^2 + 19695902139765964312200 x \\
& + 43293395755479451081500 \pi x + 4923975534941491078050 \pi) / \\
& (84134674399363759584000 x^9 + 553282870240523552976000 x^8 \\
& + 1672829200376220946752000 x^7 + 3054382387591966022592000 x^6 \\
& + 3716186788188474821020800 x^5 + 3130143936125341727592000 x^4 \\
& + 1830048825853136096208000 x^3 + 718894840186184918256000 x^2 \\
& + 173173583021917804326000 x + 19695902139765964312200) \\
F_{20} := & (93517388532372477911040 x^{10} + 29767510372010271638400 \pi x^{10} \\
& + 611837571653905732819200 x^9 + 213704525241185361768000 \pi x^9 \\
& + 711754272609689789172000 \pi x^8 + 1838400348330919552608000 x^8 \\
& + 1448837878764208396464000 \pi x^7 + 3331932215851618841404800 x^7 \\
& + 1997640105568832789064000 \pi x^6 + 4018614468781062200450400 x^6 \\
& + 1951540718517244340085600 \pi x^5 + 3350230269356550169830240 x^5 \\
& + 1370176226255545566858000 \pi x^4 + 1935035999696305265760000 x^4 \\
& + 749135365596712467172800 x^3 + 684173442749765076108000 \pi x^3 \\
& + 233240946391965366855000 \pi x^2 + 177263119257893678809800 x^2 \\
& + 19695902139765964312200 x + 49239755349414910780500 \pi x \\
& + 4923975534941491078050 \pi) / (119070041488041086553600 x^{10} \\
& + 854818100964741447072000 x^9 + 2847017090438759156688000 x^8 \\
& + 5795351515056833585856000 x^7 + 7990560422275331156256000 x^6 \\
& + 7806162874068977360342400 x^5 + 5480704905022182267432000 x^4 \\
& + 2736693770999060304432000 x^3 + 932963785567861467420000 x^2 \\
& + 196959021397659643122000 x + 19695902139765964312200)
\end{aligned} \tag{2.10}$$

In many cases the continued fraction converges on a much larger interval than the Maclaurin series, but it's not always the case.

```

> with(plots):
  display([seq](plot(F[n]-f0(x),x=-2..20,-0.01..0.01,title=
    ('n'=n)),n=1..20),insequence=true,axes=box);

```



In this case it looks like it converges for $x > -1$, but at $x = -1$ something strange happens (it's really rather a coincidence that $x = -1$ was the left endpoint of the interval I arbitrarily chose). The denominators of every fourth convergent seems to be 0 there. I don't know why.

```
> seq(eval(denom(F[n]),x=-1),n=1..20);
4, 0, 4, 4, 24, 0, 72, 36, 6660, 0, 11100, 100, 64400, 0, 450800, 4900, 167712300, 0,
301882140, 79380
```

(2.11)

The convergents are a special case of Padé approximants.

```
> F[4];
```

$$\frac{\pi x^2 + 3x^2 + \frac{3}{2}\pi x + 3x + \frac{3}{4}\pi}{4x^2 + 6x + 3}$$

(2.12)

```
> with(numapprox):
> pade(f0(x),x=0,[2,2]);
```

$$\frac{3}{4} \frac{\pi + (4 + 2\pi)x + \left(4 + \frac{4}{3}\pi\right)x^2}{4x^2 + 6x + 3}$$

(2.13)

```
> normal(%-F[4]);
```

(2.14)

$$0 \tag{2.14}$$

```
> F[5]=pade(f0(x),x=0,[3,2]);
```

$$\frac{x^3 + \frac{9}{2} \pi x^2 + 12 x^2 + \frac{27}{4} \pi x + 15 x + \frac{15}{4} \pi}{18 x^2 + 27 x + 15} \tag{2.15}$$

$$= \frac{1}{12} \frac{15 \pi + (60 + 27 \pi) x + (48 + 18 \pi) x^2 + 4 x^3}{6 x^2 + 9 x + 5}$$

```
> normal(lhs(%)-rhs(%));
```

$$0 \tag{2.16}$$

```
> F[6] = pade(f0(x),x=0,[3,3]);
```

$$\frac{6 \pi x^3 + 19 x^3 + \frac{27}{2} \pi x^2 + 30 x^2 + \frac{45}{4} \pi x + 15 x + \frac{15}{4} \pi}{24 x^3 + 54 x^2 + 45 x + 15} \tag{2.17}$$

$$= \frac{5}{4} \frac{\pi + (4 + 3 \pi) x + \left(8 + \frac{18}{5} \pi\right) x^2 + \left(\frac{76}{15} + \frac{8}{5} \pi\right) x^3}{8 x^3 + 18 x^2 + 15 x + 5}$$

```
> normal(lhs(%)-rhs(%));
```

$$0 \tag{2.18}$$

Padé meets Chebyshev

Much the same idea that takes Taylor series to Padé approximants can be used with Chebyshev series, giving us a Chebyshev-Padé approximant. Given the Chebyshev series for a function $f(x)$ on the interval $[-1,1]$, we look for a rational function $\frac{p(x)}{q(x)}$ with p and q of given degrees m and n so that the quotient agrees with the Chebyshev series for as many terms as possible. This can be done with p and q expressed in terms of Chebyshev polynomials. We'll take a partial sum

$f_r(x) = \sum_{j=0}^r c_j T(j, x)$ of the Chebyshev series for $f(x)$, and we'll want to determine coefficients a_j

and b_j so $p(x) = \sum_{j=0}^m a_j T(j, x)$ and $q(x) = \sum_{j=0}^n b_j T(j, x)$. I'll "normalize" this so $b_0 = 1$.

Thus we have $m + n + 1$ unknowns, so in general we want $m + n + 1$ equations: the coefficients of $T(j, x)$ in $\frac{p(x)}{q(x)} - f_r(x)$ should be 0 for j from 0 to $m + n$.

This is what the **chebpade** command will do, if instead of an integer n you give it a list $[m, n]$.

For example, I'll try the case $m = 8, n = 8$ for $f_0(x)$ on $[-1,1]$.

```
> Digits:= 20:
```

```
> chebpade(f0(x),x=-1..1,[8,8]);
```

```
(1.2150910857455277020 T(0, x) + 2.1504323442622673679 T(1, x)
+ 1.4817400805231678971 T(2, x) + 0.79195153209656138406 T(3, x)
```

```

+ 0.32280529047219130940 T(4, x) + 0.097736838340783278521 T(5, x)
+ 0.020806016823725443870 T(6, x) + 0.0028022649006158375805 T(7, x)
+ 0.00018045093239485958314 T(8, x)) / (T(0, x) + 1.7501206386219160280 T(1,
x) + 1.1813904164852913359 T(2, x) + 0.61171122002050111301 T(3, x)
+ 0.24087544231736790398 T(4, x) + 0.070125135833745748472 T(5, x)
+ 0.014360853134940139064 T(6, x) + 0.0018561537183627620354 T(7, x)
+ 0.00011495745645166114404 T(8, x))

```

```
> ChebPadeApp := eval(%, T=orthopoly[T]);
```

```

ChebPadeApp := (0.24364608537218874524 x + 0.74977171025922780196 x2
+ 1.3699961960050668703 x3 + 1.2499357445835586473 x5
+ 0.61959709966613015056 x6 + 0.17934495363941360515 x7
+ 1.6126256654218867027 x4 + 0.023097719346542026642 x8
+ 0.035530729803220530013) / (0.25261958170060209711 x
+ 0.69059401225410878655 x2 + 1.1482867716354041566 x3
+ 0.91411295688330262764 x5 + 0.43011819146645919718 x6
+ 0.11879383797521677027 x7 + 1.2560757810940823398 x4
+ 0.014714554425812626437 x8 + 0.045239130153588090160)

```

The terms of the Chebyshev series of **ChebPadeApp** and $f_0(x)$ should agree up to the coefficient of $T(16, x)$.

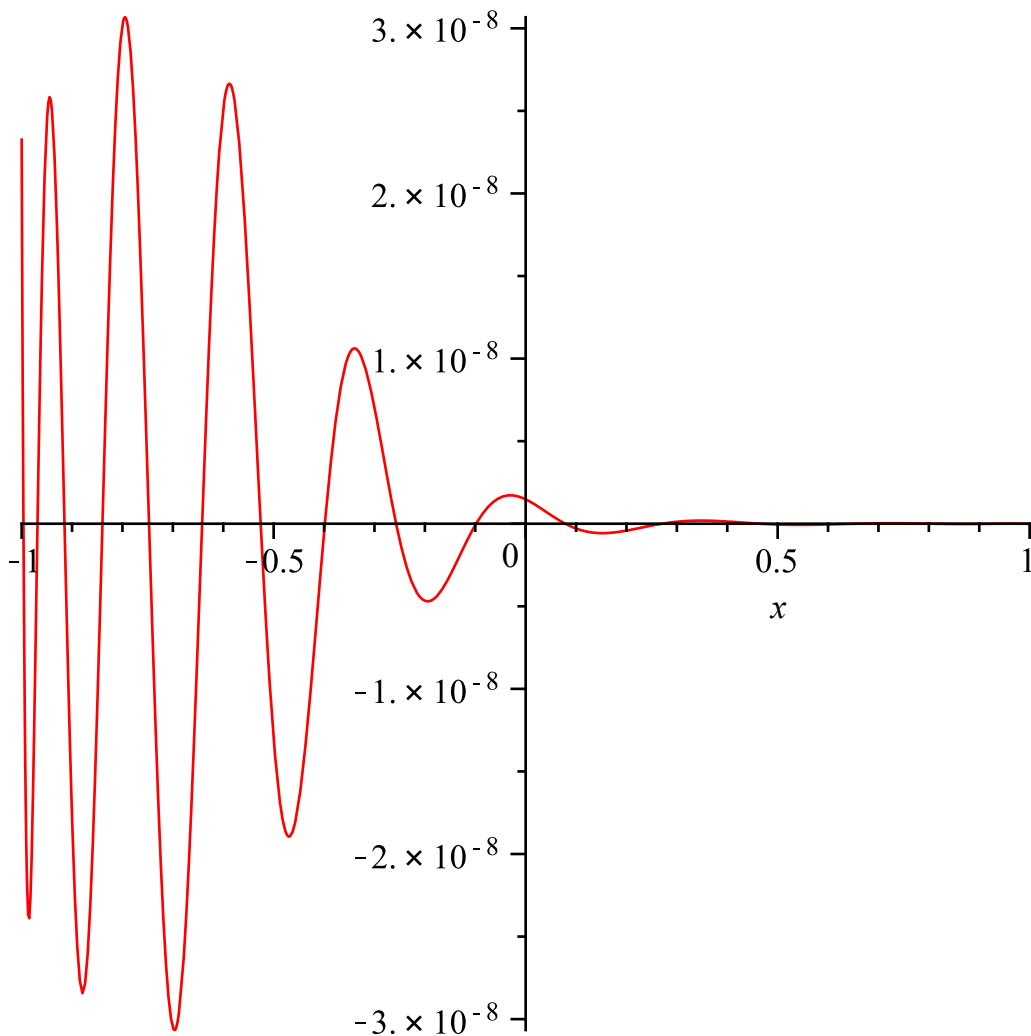
```

> chebpade(ChebPadeApp, x=-1..1, 18) - chebpade(f0, x=-1..1, 18);
-6.87 10-18 T(0, x) + 1.19 10-17 T(1, x) - 8.39 10-18 T(2, x) + 4.907 10-18 T(3, x)
- 2.193 10-18 T(4, x) + 3.67 10-19 T(5, x) + 4.796 10-19 T(6, x) - 5.097 10-19 T(7, x)
+ 1.902 10-19 T(8, x) + 5.87 10-20 T(9, x) - 1.0745 10-19 T(10, x) + 4.614 10-20 T(11,
x) + 9.83 10-21 T(12, x) - 2.158 10-20 T(13, x) + 8.8706 10-21 T(14, x)
+ 2.242 10-21 T(15, x) - 3.794 10-21 T(16, x) - 9.286321456044380 10-10 T(17, x)
+ 4.0582336243408552 10-9 T(18, x)

```

To see how well that approximates f_0 :

```
> plot(ChebPadeApp - f0(x), x=-1..1);
```

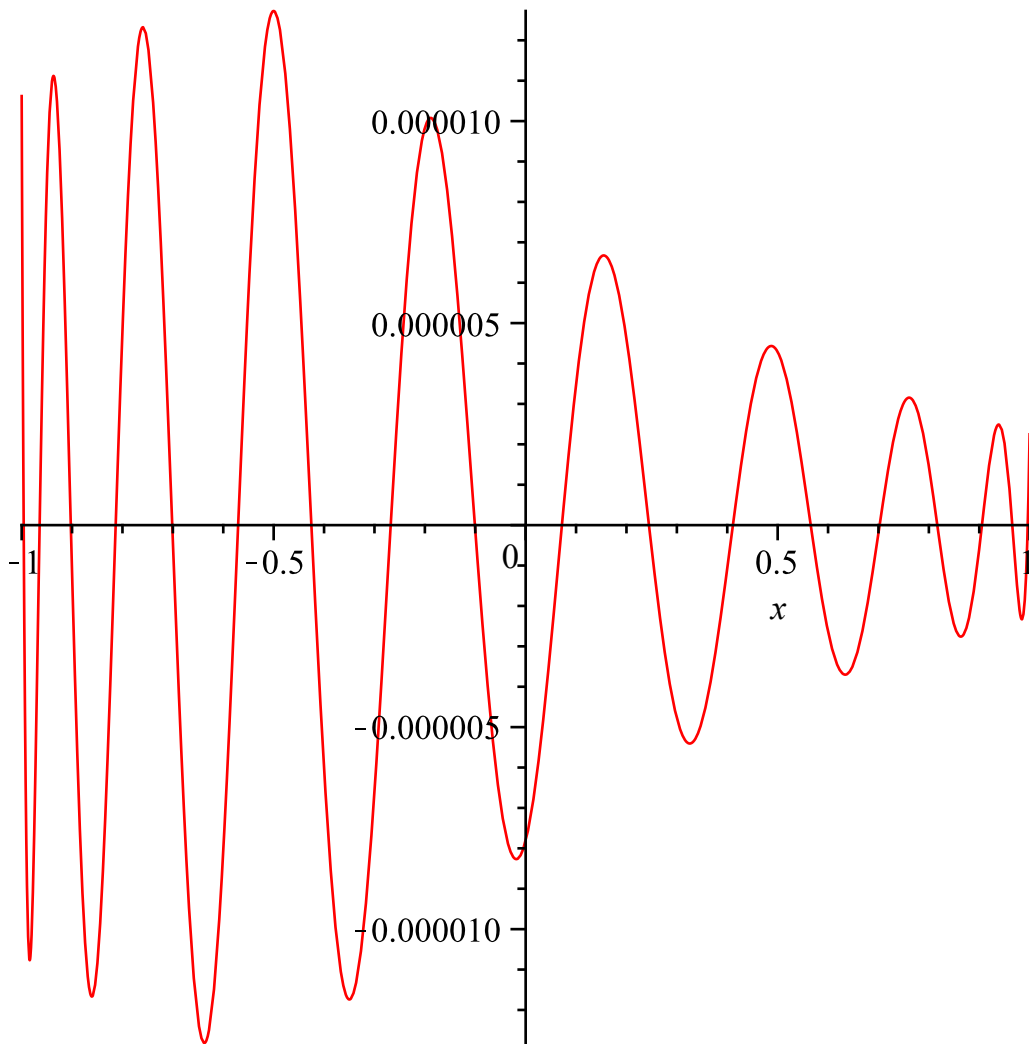


For comparison, here was the Chebyshev approximation for degree 16.

```
> ChebApp:= eval(chebpade(f0,x=-1..1,16),T=orthopoly[T]);
ChebApp:= 1.0000485515945650539 x - 0.99876822575371845519 x2
+ 0.66287353028368720111 x3 - 0.71860959386426494656 x5
+ 1.6135571716378433150 x6 - 1.8987869235996306835 x7
+ 5.3533094340438756792 x9 - 2.1110086738601142540 x10
- 0.03097264020437684917 x4 - 1.0702457755879091488 x8
+ 4.1362533555371943228 x12 - 5.8072197456164019734 x11
+ 3.0948964712226572853 x13 - 2.7785539083838447003 x14
- 0.66929394203855460141 x15 + 0.68617861519849349120 x16
+ 0.78539035222619927696
```

```
> plot(ChebApp-f0(x),x=-1..1);
```

(3.1)



Actually, we wanted an error no more than 10^{-8} so we're not quite there. Try $m=9, n=9$

```
> chebpade(f0(x), x=-1..1, [9, 9]);
(1.2204904146594178764 T(0, x) + 2.1867347687984828909 T(1, x)
+ 1.5676536778231649570 T(2, x) + 0.89619188118190663767 T(3, x)
+ 0.40421409894216092858 T(4, x) + 0.14139700264338726277 T(5, x)
+ 0.037152466295687332144 T(6, x) + 0.0069393607830995464687 T(7, x)
+ 0.00082506897523598650488 T(8, x) + 0.000047226845763143514642 T(9, x)) /
(T(0, x) + 1.7784202742264716937 T(1, x) + 1.2541098728393919652 T(2, x)
+ 0.69915932126053711718 T(3, x) + 0.30627513053055806266 T(4, x)
+ 0.10365855236204522728 T(5, x) + 0.026323861808408856132 T(6, x)
+ 0.0047443761428804202794 T(7, x) + 0.00054450546215046541514 T(8, x)
+ 0.000030057454735265050515 T(9, x))
```

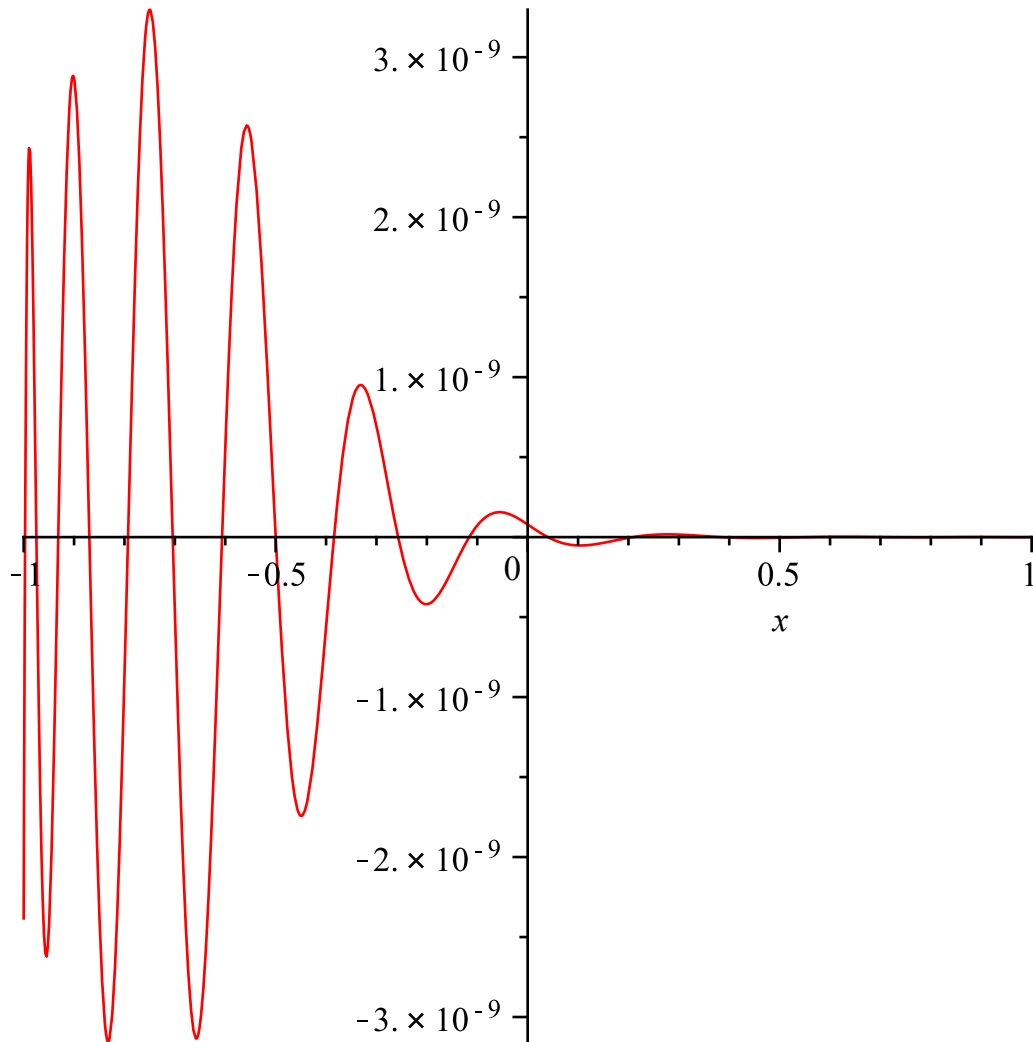
(3.2)

```
> ChebPadeApp := eval(%, T=orthopoly[T]);
ChebPadeApp := (0.15699365459987075810 x + 0.54393675022386289583 x^2
+ 1.1397644542218786757 x^3 + 1.5055456319567249981 x^5
+ 0.97766126380158208335 x^6 + 0.41691642695880030957 x^7
```

(3.3)

$$\begin{aligned}
 &+ 0.012090072515364739748 x^9 + 1.5824054453820533265 x^4 \\
 &+ 0.10560882883020627262 x^8 + 0.020723438457962502341) / \\
 &(0.16629495634754092209 x + 0.51442403919686394620 x^2 \\
 &+ 0.98554440723431565264 x^3 + 1.1401515302357510670 x^5 \\
 &+ 0.70297017955856424994 x^6 + 0.28632697921683422878 x^7 \\
 &+ 0.0076947084122278529318 x^9 + 1.2737765513849138734 x^4 \\
 &+ 0.069696699155259573138 x^8 + 0.026385901344907706743)
 \end{aligned}$$

```
> plot(ChebPadeApp-f0(x), x=-1..1);
```



Maple commands introduced in this lesson:

quotients option for cfrac

periodic option for cfrac

chebpade(..., ..., [m,n]) in numapprox package

