Lesson 34: Padé meets Chebyshev

```
> restart;
  with(numtheory):
```

Continued fractions for numbers

Consider any positive number x. We can represent it by a simple continued fraction with integer elements as follows. Let $b_0 = \text{floor}(x)$, so $0 \le x - b_0 < 1$. If that is 0, then $x = b_0$. Otherwise $x = b_0 + \frac{1}{x_1}$ with $x_1 > 1$. Let $b_1 = \text{floor}(x_1) \ge 1$. Again, if $b_1 = x_1$, then $x = b_0 + \frac{1}{b_1}$, otherwise $x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{x_2}}$ etc.

The continued fraction for x terminates if and only if x is a rational number. For example:

```
> cfrac([3,7,110,3]); CFRAC([3,7,110,3]);
```

$$\frac{7291}{2320}$$

$$3 + \cfrac{1}{7 + \cfrac{1}{110 + \cfrac{1}{3}}} \tag{1.1}$$

For an irrational number, the continued fraction does not terminate. It is eventually periodic if and only if x is a quadratic irrational, i.e. an irrational root of a quadratic polynomial with integer coefficients.

```
> cfrac(sqrt(3));
```

$$1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}}}}}}} \tag{1.2}$$

```
> cfrac(sqrt(7)/5);
```

(1.3)

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{13+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{13+\ldots}}}}}}}}}} \qquad \textbf{(1.3)}$$

You can get as many terms as you want.

```
> cfrac(sqrt(7)/5,15);
```

$$1 \Big/ \left(1 \right. \qquad \textbf{(1.4)}$$

$$+\,1 \Big/ \left(1 \right.$$

$$+\cfrac{1}{8+\cfrac{1}{13+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{13+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{8+\ldots}}}}}}}}}}}}}\Bigg)\Bigg)$$

For a more compact representation, use the quotients option.

```
> cfrac(sqrt(7)/5,30,quotients);
```

$$[0, 1, 1, 8, 13, 8, 1, 2, 1, 8, 13, 8, 1, 2, 1, 8, 13, 8, 1, 2, 1, 8, 13, 8, 1, 2, 1, 8, 13, 8, 1, \ldots] \quad \textbf{(1.5)}$$

For a quadratic irrational, you can ask cfrac to give you the initial part and the repeating part.

```
> cfrac(sqrt(7)/5,periodic,quotients);
```

$$[[0, 1], [1, 8, 13, 8, 1, 2]] \quad \textbf{(1.6)}$$

For non-quadratic irrationals, not much can be said. There are very few for which much is known.

```
> cfrac(2^(1/3),quotients,20);
```

$$[1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots] \quad \textbf{(1.7)}$$

If you could prove that the sequence of quotients here (or for any non-quadratic algebraic number) is unbounded, or that it is bounded, you would become rather famous. The same for these:

```
> cfrac(Pi,quotients,20);
```

$$[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, \ldots] \quad \textbf{(1.8)}$$

```
> cfrac(gamma,quotients,20);
```

$$[0, 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, 1, \ldots] \quad \textbf{(1.9)}$$

Some do have known patterns:

```
> cfrac(exp(1),quotients,20);
```

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots] \quad \textbf{(1.10)}$$

```
> cfrac(tan(1),quotients,20);
```

$$[1, 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, 13, 1, 15, 1, 17, 1, 19, 1, \ldots] \quad \textbf{(1.11)}$$

The convergents of the continued fraction provide, in certain senses, the best rational approximations of an irrational number.

- For any positive integers a and b with $b \le Q_n$, $\left|x - \frac{a}{b}\right| \ge \left|x - \frac{P_n}{Q_n}\right|$
- For each n, $\left|x - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2}$, and at least one of $\left|x - \frac{P_n}{Q_n}\right| < \frac{1}{2\,Q_n^2}$ or $\left|x - \frac{P_{n+1}}{Q_{n+1}}\right| < \frac{1}{2\,Q_{n+1}^2}$
- Any rational

- $\frac{a}{b}$ in lowest terms with $\left|x - \frac{a}{b}\right| < \frac{1}{2\,b^2}$ is a convergent of x.

So e.g. for π we get the following good approximations:

```
> S:= [seq(nthconver(cfrac(Pi,5),n),n=1..5)];
```

$$S := \left[\frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}\right] \quad \textbf{(1.12)}$$

```
> Digits:= 15; seq(evalf(abs(Pi - S[n])),n=1..5);
```

$$Digits := 15$$

$$0.00126448926735, 0.00008321962753, 2.6676419\ 10^{-7}, 5.7789\ 10^{-10}, 3.3163\ 10^{-10} \quad \textbf{(1.13)}$$

```
> seq(evalf(abs(Pi-S[n])*denom(S[n])^2), n=1..5);
```

$$0.061959974100, 0.9350557349, 0.0034063120, 0.63322, 0.36587 \quad \textbf{(1.14)}$$

Continued fractions for analytic functions

Consider again our function f0.

```
> f0:= x -> arctan(2*x+1);
```

$$f0 := x \rightarrow \arctan(2\,x+1) \quad \textbf{(2.1)}$$

```
> Q:=cfrac(f0(x),x,21,quotients);
```

$$Q := \left[\frac{1}{4}\,\pi, [x, 1], [x, 1], [-x, 3], [4\,x, 1], [-x, 5], [6\,x, 1], [-3\,x, 14], [17\,x, 1], [-32\,x, 153], [185\,x, 1], [-85\,x, 407], [492\,x, 1], [-111\,x, 533], [644\,x, 1], [-287\,x, 1380], [1667\,x, 1], [-5888\,x, 28339], [34227\,x, 1], [-15003\,x, 72257], [87260\,x, 1], [-19015\,x, 91623], \ldots\right] \quad \textbf{(2.2)}$$

```
> CFRAC(Q[1..5]);
```

$$\frac{1}{4}\,\pi + \cfrac{x}{1 + \cfrac{x}{1 - \cfrac{x}{3 + \cfrac{4\,x}{1}}}} \quad \textbf{(2.3)}$$

How do these come about? In a way it's rather analogous to what we had for numbers.

```
> f0(0);
```

$$\frac{1}{4}\,\pi \quad \textbf{(2.4)}$$

```
> f1:= solve(f0(x) = Pi/4 + x/f1,f1); series(f1, x);
```

$$f1 := \frac{4\,x}{4\arctan(2\,x+1) - \pi}$$

$$1 + x + \frac{1}{3}\,x^2 - \frac{1}{3}\,x^3 + \frac{11}{45}\,x^4 + O(x^5) \quad \textbf{(2.5)}$$

```
> f2:= solve(f1 = 1 + x/f2, f2); series(f2,x);
```

$$f2 := -\frac{x\left(4\arctan(2x+1)-\pi\right)}{-4x+4\arctan(2x+1)-\pi}$$

$$1-\frac{1}{3}x+\frac{4}{9}x^2-\frac{68}{135}x^3+O(x^4) \quad \textbf{(2.6)}$$

```
> f3:= solve(f2 = 1 + x/f3, f3); series(f3,x);
```

$$f3 := -\frac{x\left(-4x+4\arctan(2x+1)-\pi\right)}{4x\arctan(2x+1)-x\pi-4x+4\arctan(2x+1)-\pi}$$

$$-3-4x-\frac{4}{5}x^2+O(x^3) \quad \textbf{(2.7)}$$

```
> f4:= solve(f3 = -3 + x/f4, f4); series(f4,x);
```

$$f4 := \frac{x\left(4x\arctan(2x+1)-x\pi-4x+4\arctan(2x+1)-\pi\right)}{4x^2+8x\arctan(2x+1)-2x\pi-12x+12\arctan(2x+1)-3\pi}$$

$$-\frac{1}{4}+\frac{1}{20}x+O(x^2) \quad \textbf{(2.8)}$$

And thus

$$f0(x) = \frac{\pi}{4}+\frac{x}{f1} = \frac{\pi}{4}+\frac{x}{1+\frac{x}{f2}} = \frac{\pi}{4}+\frac{x}{1+\frac{x}{1+\frac{x}{f3}}} = \frac{\pi}{4}+\frac{x}{1+\frac{x}{1+\frac{x}{-3+\frac{x}{f4}}}}$$

$$= \frac{\pi}{4}+\frac{x}{1+\frac{x}{1+\frac{x}{-3+\frac{x}{-\frac{1}{4}+\frac{x}{\dots}}}}}$$

which Maple prefers to write as

```
> CFRAC([(1/4)*Pi, [x, 1], [x, 1], [-x, 3], [4*x, 1],`...`]);
```

$$\frac{1}{4}\pi+\frac{x}{1+\frac{x}{1-\frac{x}{3+\frac{4x}{1+\dots}}}} \quad \textbf{(2.9)}$$

```
> for nn from 1 to 20 do F[nn]:= nthconver(Q,nn) end do;
```

$$F_1 := x+\frac{1}{4}\pi$$

$$F_2 := \frac{\frac{1}{4}\pi x+x+\frac{1}{4}\pi}{x+1}$$

$$F_3 := \frac{-x^2+\frac{1}{2}\pi x+3x+\frac{3}{4}\pi}{2x+3}$$

$$F_4 := \frac{\pi x^2 + 3 x^2 + \frac{3}{2} \pi x + 3 x + \frac{3}{4} \pi}{4 x^2 + 6 x + 3}$$

$$F_5 := \frac{x^3 + \frac{9}{2} \pi x^2 + 12 x^2 + \frac{27}{4} \pi x + 15 x + \frac{15}{4} \pi}{18 x^2 + 27 x + 15}$$

$$F_6 := \frac{6 \pi x^3 + 19 x^3 + \frac{27}{2} \pi x^2 + 30 x^2 + \frac{45}{4} \pi x + 15 x + \frac{15}{4} \pi}{24 x^3 + 54 x^2 + 45 x + 15}$$

$$F_7 := \frac{-3 x^4 + \frac{141}{2} \pi x^3 + 230 x^3 + \frac{675}{4} \pi x^2 + 375 x^2 + \frac{585}{4} \pi x + 210 x + \frac{105}{2} \pi}{282 x^3 + 675 x^2 + 585 x + 210}$$

$$F_8 := \frac{1}{408 x^4 + 1200 x^3 + 1440 x^2 + 840 x + 210} \Big(320 x^4 + 102 \pi x^4 + 300 \pi x^3 + 740 x^3 + 360 \pi x^2 + 630 x^2 + 210 x + 210 \pi x + \frac{105}{2} \pi \Big)$$

$$F_9 := \Big(96 x^5 + 41600 x^4 + 13350 \pi x^4 + 101220 x^3 + 40500 \pi x^3 + 89670 x^2 + 50400 \pi x^2 + 30450 \pi x + 32130 x + \frac{16065}{2} \pi \Big) \Big/ \left(53400 x^4 + 162000 x^3 + 201600 x^2 + 121800 x + 32130 \right)$$

$$F_{10} := \Big(59296 x^5 + 18870 \pi x^5 + 178500 x^4 + 68850 \pi x^4 + 217770 x^3 + 107100 \pi x^3 + 128520 x^2 + 89250 \pi x^2 + \frac{80325}{2} \pi x + 32130 x + \frac{16065}{2} \pi \Big) \Big/ \left(75480 x^5 + 275400 x^4 + 428400 x^3 + 357000 x^2 + 160650 x + 32130 \right)$$

$$F_{11} := \Big(-8160 x^6 + 20597472 x^5 + 6545340 \pi x^5 + 64045800 x^4 + 24579450 \pi x^4 + 39305700 \pi x^3 + 81010440 x^3 + 49576590 x^2 + 33736500 \pi x^2 + 13076910 x + 15663375 \pi x + \frac{6538455}{2} \pi \Big) \Big/ \left(26181360 x^5 + 98317800 x^4 + 157222800 x^3 + 134946000 x^2 + 62653500 x + 13076910 \right)$$

$$F_{12} := \Big(9284040 \pi x^6 + 29165472 x^6 + 40419540 \pi x^5 + 108419472 x^5 + 171188640 x^4 + 77272650 \pi x^4 + 144242280 x^3 + 83216700 \pi x^3 + 65384550 x^2 + 53496450 \pi x^2 + 19615365 \pi x + 13076910 x + \frac{6538455}{2} \pi \Big) \Big/ \left(37136160 x^6 + 161678160 x^5 + 309090600 x^4 + 332866800 x^3 + 213985800 x^2 + 78461460 x + 13076910 \right)$$

$$F_{13} := \left(905760\,x^7 + 4221860580\,\pi\,x^6 + 13258877184\,x^6 + 50678494776\,x^5 + 18815295870\,\pi\,x^5 + 82251386280\,x^4 + 36823389750\,\pi\,x^4 + 40609749600\,\pi\,x^3 + 71378133750\,x^3 + 33398428140\,x^2 + 26774973225\,\pi\,x^2 + 6969993030\,x + \frac{20184210585}{2}\,\pi\,x + \frac{3484996515}{2}\,\pi\right) \Big/ \left(16887442320\,x^6 + 75261183480\,x^5 + 147293559000\,x^4 + 162438998400\,x^3 + 107099892900\,x^2 + 40368421170\,x + 6969993030\right)$$

$$F_{14} := \left(5978921760\,\pi\,x^7 + 18783469728\,x^7 + 30252044340\,\pi\,x^6 + 83081017152\,x^6 + 160923978936\,x^5 + 68578882470\,\pi\,x^5 + 175143414600\,x^4 + 90414944550\,\pi\,x^4 + 113485783950\,x^3 + 75061463400\,\pi\,x^3 + 41819958180\,x^2 + 39407268285\,\pi\,x^2 + \frac{24394975605}{2}\,\pi\,x + 6969993030\,x + \frac{3484996515}{2}\,\pi\right) \Big/ \left(23915687040\,x^7 + 121008177360\,x^6 + 274315529880\,x^5 + 361659778200\,x^4 + 300245853600\,x^3 + 157629073140\,x^2 + 48789951210\,x + 6969993030\right)$$

$$F_{15} := \left(-259953120\,x^8 + 7039238042340\,\pi\,x^7 + 22115890472832\,x^7 + 36347831274510\,\pi\,x^6 + 100107075669048\,x^6 + 198468943069320\,x^5 + 84070544950350\,\pi\,x^5 + 221212387761750\,x^4 + 113117625343800\,\pi\,x^4 + 147025032974820\,x^3 + 95900402176425\,\pi\,x^3 + 55711154288790\,x^2 + \frac{102971192028705}{2}\,\pi\,x^2 + 9618590381400\,x + \frac{32664872335095}{2}\,\pi\,x + 2404647595350\,\pi\right) \Big/ \left(28156952169360\,x^7 + 145391325098040\,x^6 + 336282179801400\,x^5 + 452470501375200\,x^4 + 383601608705700\,x^3 + 205942384057410\,x^2 + 65329744670190\,x + 9618590381400\right)$$

$$F_{16} := \left(31311784083456\,x^8 + 9966862573920\,\pi\,x^8 + 57469395957120\,\pi\,x^7 + 160611946065216\,x^7 + 150668828352000\,\pi\,x^6 + 368367348555360\,x^6 + 490433015207520\,x^5 + 234792257515200\,\pi\,x^5 + 410393189606400\,x^4 + 238245084831600\,\pi\,x^4 + 161592318407520\,\pi\,x^3 + 216738903260880\,x^3 + 67330132669800\,x^2 + 71818808181120\,\pi\,x^2 + 9618590381400\,x + 19237180762800\,\pi\,x + 2404647595350\,\pi\right) \Big/ \left(39867450295680\,x^8\right.$$

$$+ 229877583828480\, x^7 + 602675313408000\, x^6 + 939169030060800\, x^5 + 952980339326400\, x^4 + 646369273630080\, x^3 + 287275232724480\, x^2 + 76948723051200\, x + 9618590381400 \big)$$

$$F_{17} := \big(1530603970560\, x^9 + 241003884889020960\, \pi\, x^8 + 757126286037024768\, x^8 + 1414609181484508800\, \pi\, x^7 + 3962151478002801600\, x^7 + 3774796557999667200\, \pi\, x^6 + 9270577153918190880\, x^6 + 5987741207698958400\, \pi\, x^5 + 12595882678824725280\, x^5 + 10764449206100029440\, x^4 + 6186965891027922000\, \pi\, x^4 + 4276217522018201760\, \pi\, x^3 + 5814136503057682800\, x^3 + 1939107820890240000\, \pi\, x^2 + 1851434369563779000\, x^2 + 531003900595568400\, \pi\, x + 272581232818494600\, x + 68145308204623650\, \pi \big) \Big/ \big(964015539556083840\, x^8 + 5658436725938035200\, x^7 + 15099186231998668800\, x^6 + 23950964830795833600\, x^5 + 24747863564111688000\, x^4 + 17104870088072807040\, x^3 + 7756431283560960000\, x^2 + 2124015602382273600\, x + 272581232818494600 \big)$$

$$F_{18} := \big(1071709964428419072\, x^9 + 341135805317559840\, \pi\, x^9 + 6254391364011172800\, x^8 + 2208008900313367200\, \pi\, x^8 + 6571551169488412800\, \pi\, x^7 + 16570260717007108320\, x^7 + 11811031155972417600\, \pi\, x^6 + 26056627965425977920\, x^6 + 14142155726230131600\, \pi\, x^5 + 26642410379482978080\, x^5 + 18182771648010169200\, x^4 + 11717786173162109040\, \pi\, x^4 + 8118644953946927400\, x^3 + 6734359869633396000\, \pi\, x^3 + 2180649862547956800\, x^2 + 2597538806858595600\, \pi\, x^2 + 613307773841612850\, \pi\, x + 272581232818494600\, x + 68145308204623650\, \pi \big) \Big/ \big(1364543221270239360\, x^9 + 8832035601253468800\, x^8 + 26286204677953651200\, x^7 + 47244124623889670400\, x^6 + 56568622904920526400\, x^5 + 46871144692648436160\, x^4 + 26937439478533584000\, x^3 + 10390155227434382400\, x^2 + 2453231095366451400\, x + 272581232818494600 \big)$$

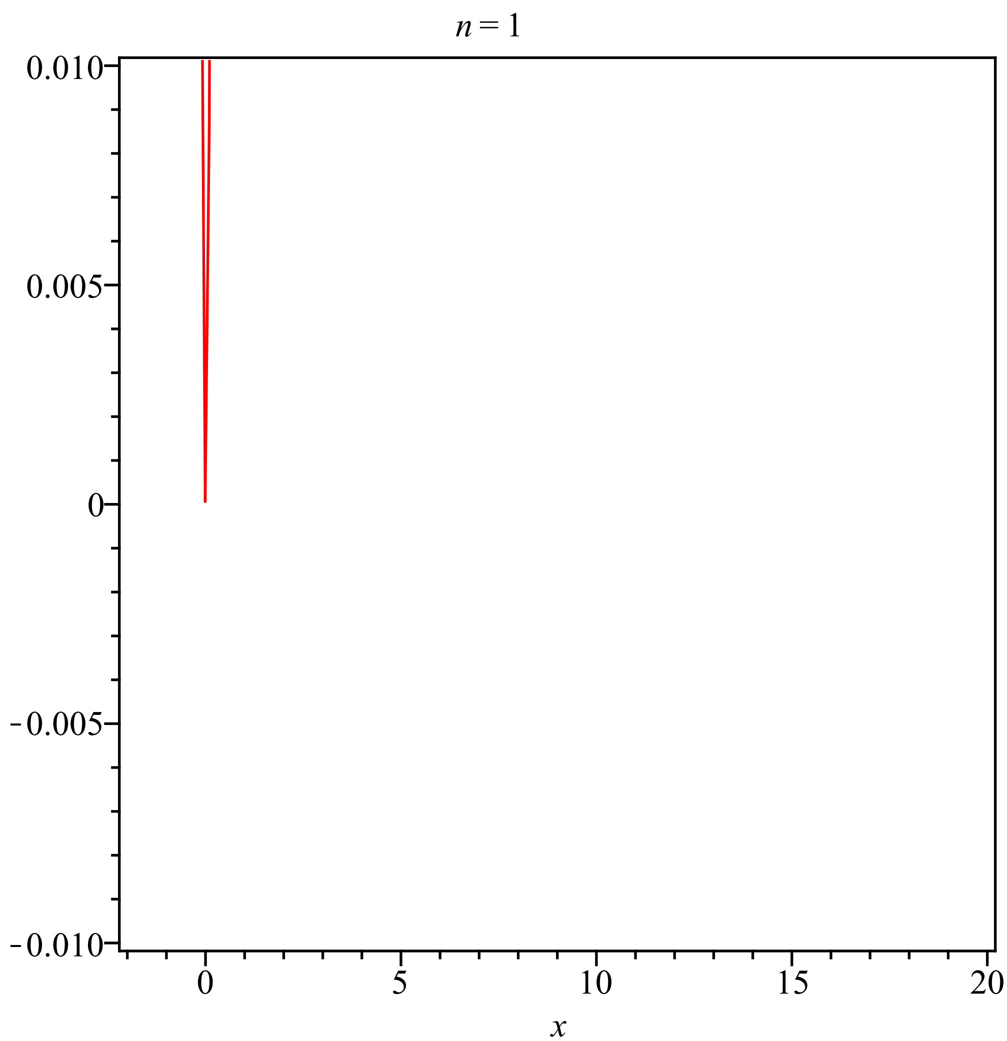
$$F_{19} := \big(-22963651370311680\, x^{10} + 660793812302907942912000\, x^9 + 2103366859984093989600\, \pi\, x^9 + 13832071756013088824400\, \pi\, x^8 + 39247939816487928060480\, x^8 + 105823085958854800810560\, x^7 + 41820730009405523668800\, \pi\, x^7 + 76359559689799150564800\, \pi\, x^6 + 169379773906737753318960\, x^6 + 92904669704711870525520\, \pi\, x^5 + 176360161535118280543824\, x^5 + 78253598403133543189800\, \pi\, x^4$$

$$+ 1226603041014896380836000\, x^{4} + 457512206463284024052000\, \pi\, x^{3}$$
$$+ 558851858590777756804800\, x^{3} + 179723710046546229564000\, \pi\, x^{2}$$
$$+ 153477680882151840013800\, x^{2} + 19695902139765964312200\, x$$
$$+ 43293395755479451081500\, \pi\, x + 4923975534941491078050\, \pi \Big) \Big/$$
$$\Big(841346743993637595840000\, x^{9} + 553282870240523552976000\, x^{8}$$
$$+ 1672829200376220946752000\, x^{7} + 3054382387591966022592000\, x^{6}$$
$$+ 3716186788188474821020800\, x^{5} + 3130143936125341727592000\, x^{4}$$
$$+ 1830048825853136096208000\, x^{3} + 718894840186184918256000\, x^{2}$$
$$+ 173173583021917804326000\, x + 19695902139765964312200 \Big)$$

$$F_{20} := \Big(93517388532372477911040\, x^{10} + 29767510372010271638400\, \pi\, x^{10} \qquad \textbf{(2.10)}$$
$$+ 611837571653905732819200\, x^{9} + 213704525241185361768000\, \pi\, x^{9}$$
$$+ 711754272609689789172000\, \pi\, x^{8} + 1838400348330919552608000\, x^{8}$$
$$+ 1448837878764208396464000\, \pi\, x^{7} + 3331932215851618841404800\, x^{7}$$
$$+ 1997640105568832789064000\, \pi\, x^{6} + 4018614468781062200450400\, x^{6}$$
$$+ 1951540718517244340085600\, \pi\, x^{5} + 3350230269356550169830240\, x^{5}$$
$$+ 1370176222625554556685800\, \pi\, x^{4} + 1935035999696305265760000\, x^{4}$$
$$+ 749135365596712467172800\, x^{3} + 684173442749765076108000\, \pi\, x^{3}$$
$$+ 233240946391965366855000\, \pi\, x^{2} + 177263119257893678809800\, x^{2}$$
$$+ 19695902139765964312200\, x + 49239755349414910780500\, \pi\, x$$
$$+ 4923975534941491078050\, \pi \Big) \Big/ \Big(119070041488041086553600\, x^{10}$$
$$+ 854818100964741447072000\, x^{9} + 2847017090438759156688000\, x^{8}$$
$$+ 5795351515056833585856000\, x^{7} + 7990560422275331156256000\, x^{6}$$
$$+ 7806162874068977360342400\, x^{5} + 5480704905022182267432000\, x^{4}$$
$$+ 2736693770999060304432000\, x^{3} + 932963785567861467420000\, x^{2}$$
$$+ 196959021397659643122000\, x + 19695902139765964312200 \Big)$$

In many cases the continued fraction converges on a much larger interval than the Maclaurin series, but it's not always the case.

```
> with(plots):
  display([seq](plot(F[n]-f0(x),x=-2..20,-0.01..0.01,title=
  ('n'=n)),n=1..20),insequence=true,axes=box);
```



In this case it looks like it converges for $x > -1$, but at $x = -1$ something strange happens (it's really rather a coincidence that $x = -1$ was the left endpoint of the interval I arbitrarily chose). The denominators of every fourth convergent seems to be 0 there. I don't know why.

```
> seq(eval(denom(F[n]),x=-1),n=1..20);
```

4, 0, 4, 4, 24, 0, 72, 36, 6660, 0, 11100, 100, 64400, 0, 450800, 4900, 167712300, 0, 301882140, 79380 **(2.11)**

The convergents are a special case of Padé approximants.

```
> F[4];
```

$$\frac{\pi x^2 + 3 x^2 + \frac{3}{2} \pi x + 3 x + \frac{3}{4} \pi}{4 x^2 + 6 x + 3} \tag{2.12}$$

```
> with(numapprox):
> pade(f0(x),x=0,[2,2]);
```

$$\frac{3}{4} \frac{\pi + (4 + 2\pi) x + \left(4 + \frac{4}{3}\pi\right) x^2}{4 x^2 + 6 x + 3} \tag{2.13}$$

```
> normal(%-F[4]);
```

(2.14)

$$0 \quad (2.14)$$

```
> F[5]=pade(f0(x),x=0,[3,2]);
```

$$\frac{x^3 + \frac{9}{2}\pi x^2 + 12 x^2 + \frac{27}{4}\pi x + 15 x + \frac{15}{4}\pi}{18 x^2 + 27 x + 15} = \frac{1}{12} \frac{15\pi + (60 + 27\pi) x + (48 + 18\pi) x^2 + 4 x^3}{6 x^2 + 9 x + 5} \quad (2.15)$$

```
> normal(lhs(%)-rhs(%));
```

$$0 \quad (2.16)$$

```
> F[6] = pade(f0(x),x=0,[3,3]);
```

$$\frac{6\pi x^3 + 19 x^3 + \frac{27}{2}\pi x^2 + 30 x^2 + \frac{45}{4}\pi x + 15 x + \frac{15}{4}\pi}{24 x^3 + 54 x^2 + 45 x + 15} = \frac{5}{4} \frac{\pi + (4 + 3\pi) x + \left(8 + \frac{18}{5}\pi\right) x^2 + \left(\frac{76}{15} + \frac{8}{5}\pi\right) x^3}{8 x^3 + 18 x^2 + 15 x + 5} \quad (2.17)$$

```
> normal(lhs(%)-rhs(%));
```

$$0 \quad (2.18)$$

Padé meets Chebyshev

Much the same idea that takes Taylor series to Padé approximants can be used with Chebyshev series, giving us a Chebyshev-Padé approximant. Given the Chebyshev series for a function $f(x)$ on the interval [-1,1], we look for a rational function $\frac{p(x)}{q(x)}$ with p and q of given degrees m and n so that the quotient agrees with the Chebyshev series for as many terms as possible. This can be done with p and q expressed in terms of Chebyshev polynomials. We'll take a partial sum $f_r(x) = \sum_{j=0}^{r} c_j T(j, x)$ of the Chebyshev series for $f(x)$, and we'll want to determine coefficients a_j and b_j so $p(x) = \sum_{j=0}^{m} a_j T(j, x)$ and $q(x) = \sum_{j=0}^{n} b_j T(j, x)$. I'll "normalize" this so $b_0 = 1$.

Thus we have $m + n + 1$ unknowns, so in general we want $m + n + 1$ equations: the coefficients of $T(j, x)$ in $\frac{p(x)}{q(x)} - f_r(x)$ should be 0 for j from 0 to $m + n$.

This is what the **chebpade** command will do, if instead of an integer n you give it a list $[m, n]$.

For example, I'll try the case $m = 8$, $n = 8$ for $f_0(x)$ on [-1,1].

```
> Digits:= 20:
> chebpade(f0(x),x=-1..1,[8,8]);
```

$$(1.2150910857455277020\, T(0, x) + 2.1504323442622673679\, T(1, x)$$
$$+ 1.4817400805231678971\, T(2, x) + 0.79195153209656138406\, T(3, x)$$

$+ 0.3228052904721913094 0\, T(4, x) + 0.09773683834078327852 1\, T(5, x)$
$+ 0.02080601682372544387 0\, T(6, x) + 0.00280226490061583758 05\, T(7, x)$
$+ 0.00018045093239485958314\, T(8, x))\,/\,(T(0, x) + 1.7501206386219160280\, T(1, x) + 1.1813904164852913359\, T(2, x) + 0.61171122002050111301\, T(3, x)$
$+ 0.24087544231736790398\, T(4, x) + 0.070125135833745748472\, T(5, x)$
$+ 0.014360853134940139064\, T(6, x) + 0.0018561537183627620354\, T(7, x)$
$+ 0.00011495745645166114404\, T(8, x))$

```
> ChebPadeApp:= eval(%,T=orthopoly[T]);
```

$ChebPadeApp := (0.24364608537218874524\, x + 0.74977171025922780196\, x^2 + 1.3699961960050668703\, x^3 + 1.2499357445835586473\, x^5 + 0.61959709966613015056\, x^6 + 0.17934495363941360515\, x^7 + 1.6126256654218867027\, x^4 + 0.023097719346542026642\, x^8 + 0.035530729803220530013)\,/\,(0.25261958170060209711\, x + 0.69059401225410878655\, x^2 + 1.1482867716354041566\, x^3 + 0.91411295688330262764\, x^5 + 0.43011819146645919718\, x^6 + 0.11879383797521677027\, x^7 + 1.2560757810940823398\, x^4 + 0.014714554425812626437\, x^8 + 0.045239130153588090160)$

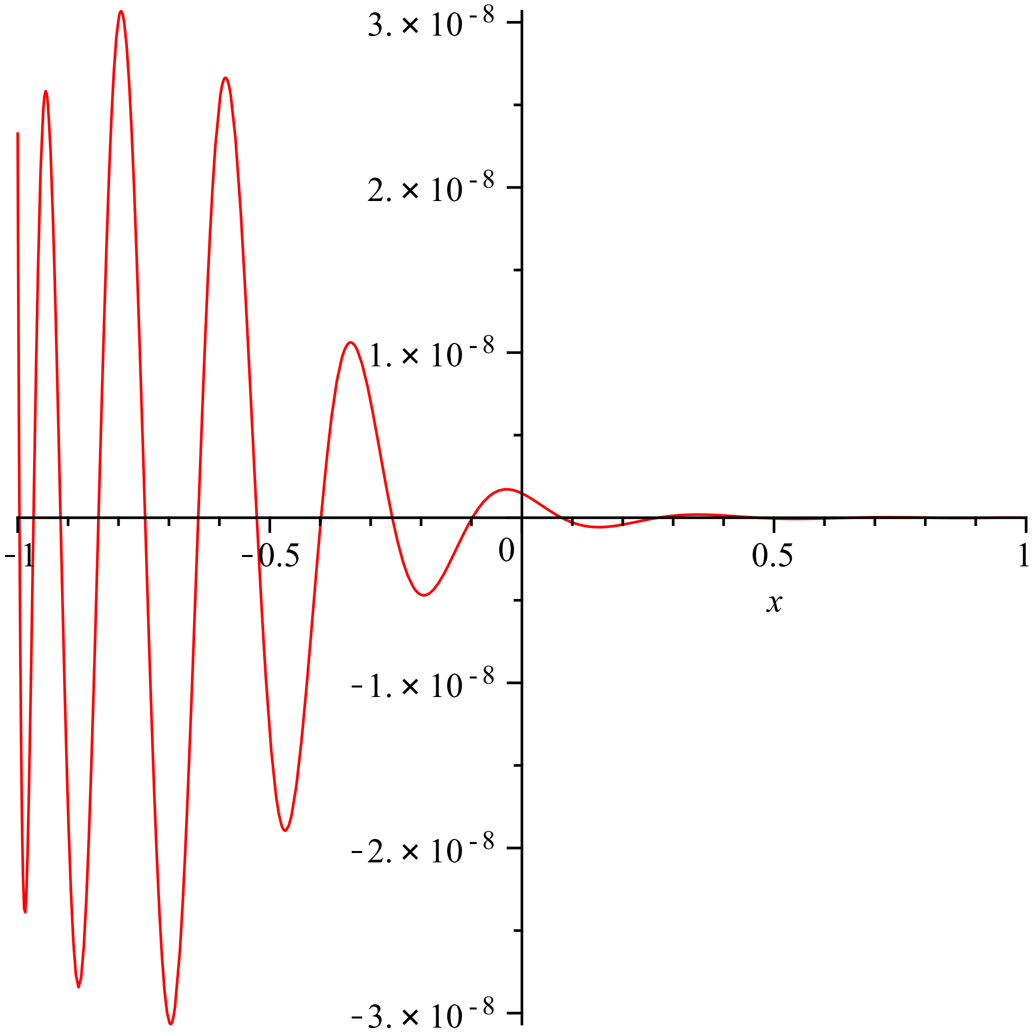
The terms of the Chebyshev series of **ChebPadeApp** and $f_0(x)$ should agree up to the coefficient of $T(16, x)$.

```
> chebpade(ChebPadeApp,x=-1..1,18)-chebpade(f0,x=-1..1,18);
```

$-6.87\ 10^{-18}\, T(0, x) + 1.19\ 10^{-17}\, T(1, x) - 8.39\ 10^{-18}\, T(2, x) + 4.907\ 10^{-18}\, T(3, x) - 2.193\ 10^{-18}\, T(4, x) + 3.67\ 10^{-19}\, T(5, x) + 4.796\ 10^{-19}\, T(6, x) - 5.097\ 10^{-19}\, T(7, x) + 1.902\ 10^{-19}\, T(8, x) + 5.87\ 10^{-20}\, T(9, x) - 1.0745\ 10^{-19}\, T(10, x) + 4.614\ 10^{-20}\, T(11, x) + 9.83\ 10^{-21}\, T(12, x) - 2.158\ 10^{-20}\, T(13, x) + 8.8706\ 10^{-21}\, T(14, x) + 2.242\ 10^{-21}\, T(15, x) - 3.794\ 10^{-21}\, T(16, x) - 9.286321456044380\ 10^{-10}\, T(17, x) + 4.0582336243408552\ 10^{-9}\, T(18, x)$

To see how well that approximates f_0:

```
> plot(ChebPadeApp-f0(x),x=-1..1);
```

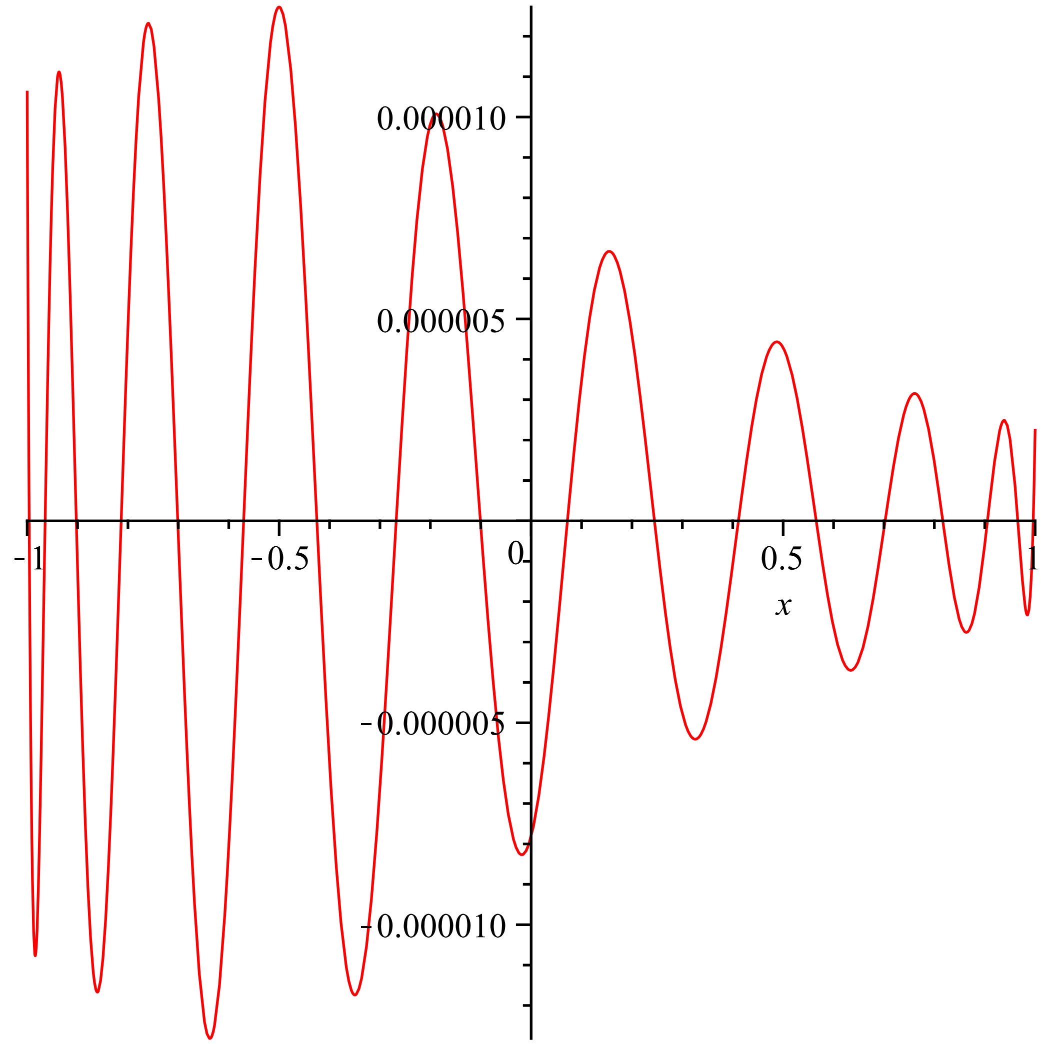


For comparison, here was the Chebyshev approximation for degree 16.

```
> ChebApp:= eval(chebpade(f0,x=-1..1,16),T=orthopoly[T]);
```

$$\begin{aligned} ChebApp := {} & 1.0000485515945650539\,x - 0.99876822575371845519\,x^2 \\ & + 0.66287353028368720111\,x^3 - 0.71860959386426494656\,x^5 \\ & + 1.6135571716378433150\,x^6 - 1.8987869235996306835\,x^7 \\ & + 5.3533094340438756792\,x^9 - 2.1110086738601142540\,x^{10} \\ & - 0.030972640204376849 17\,x^4 - 1.0702457755879091488\,x^8 \\ & + 4.1362533555371943228\,x^{12} - 5.8072197456164019734\,x^{11} \\ & + 3.0948964712226572853\,x^{13} - 2.7785539083838447003\,x^{14} \\ & - 0.66929394203855460141\,x^{15} + 0.68617861519849349120\,x^{16} \\ & + 0.78539035222619927696 \end{aligned} \quad \textbf{(3.1)}$$

```
> plot(ChebApp-f0(x),x=-1..1);
```



Actually, we wanted an error no more than 10^{-8} so we're not quite there. Try $m = 9$, $n = 9$

```
> chebpade(f0(x),x=-1..1,[9,9]);
```

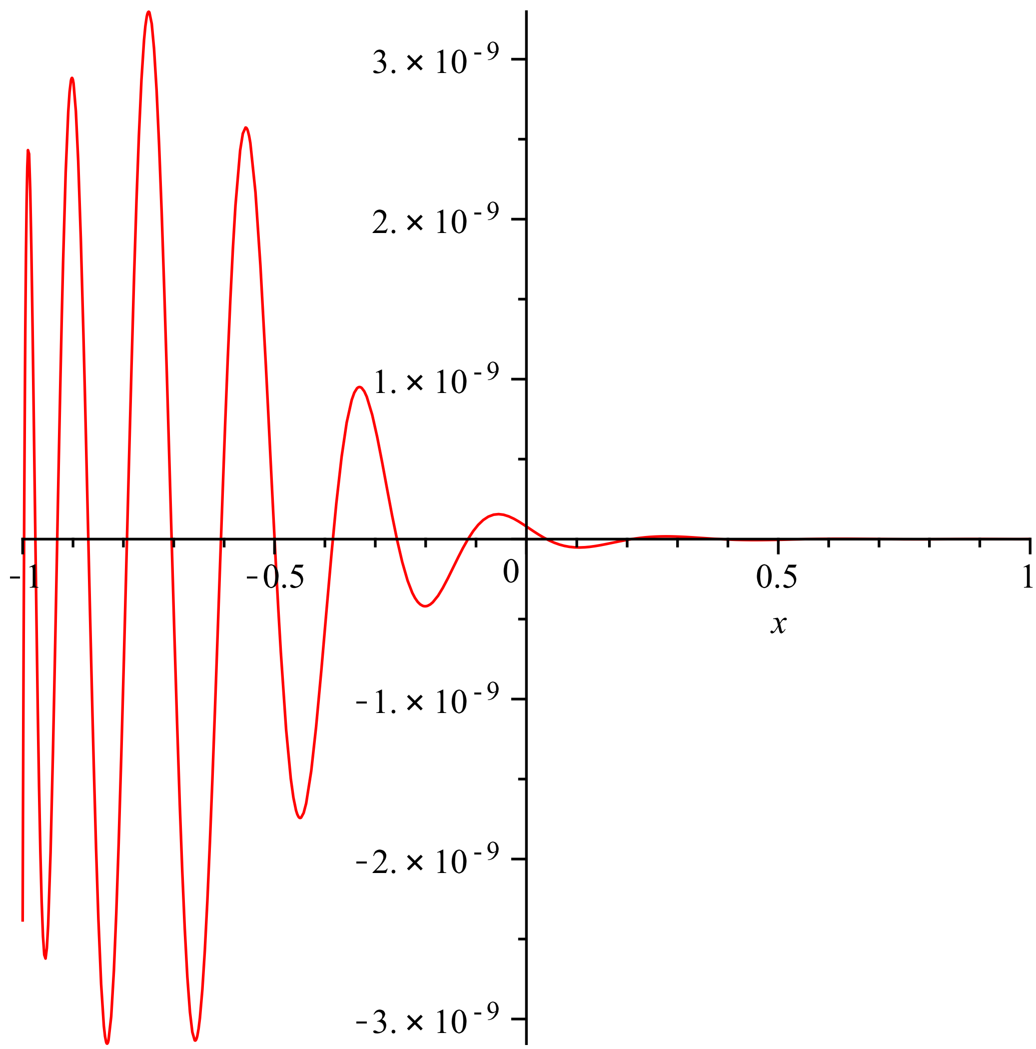
$$(1.2204904146594178764\,T(0,x) + 2.1867347687984828909\,T(1,x) + 1.5676536778231649570\,T(2,x) + 0.89619188118190663767\,T(3,x) + 0.40421409894216092858\,T(4,x) + 0.14139700264338726277\,T(5,x) + 0.037152466295687332144\,T(6,x) + 0.0069393607830995464687\,T(7,x) + 0.00082506897523598650488\,T(8,x) + 0.000047226845763143514642\,T(9,x)) / (T(0,x) + 1.7784202742264716937\,T(1,x) + 1.2541098728393919652\,T(2,x) + 0.69915932126053711718\,T(3,x) + 0.30627513053055806266\,T(4,x) + 0.10365855236204522728\,T(5,x) + 0.026323861808408856132\,T(6,x) + 0.0047443761428804202794\,T(7,x) + 0.00054450546215046541514\,T(8,x) + 0.000030057454735265050515\,T(9,x)) \tag{3.2}$$

```
> ChebPadeApp:= eval(%,T=orthopoly[T]);
```

$$ChebPadeApp := (0.15699365459987075810\,x + 0.54393675022386289583\,x^2 + 1.1397644542218786757\,x^3 + 1.5055456319567249981\,x^5 + 0.97766126380158208335\,x^6 + 0.41691642695880030957\,x^7 \tag{3.3}$$

$$+ 0.01209007251536473974 8 x^9 + 1.5824054453820533265 x^4 + 0.10560882883020627262 x^8 + 0.020723438457962502341) / (0.16629495634754092209 x + 0.51442403919686394620 x^2 + 0.98554440723431565264 x^3 + 1.1401515302357510670 x^5 + 0.70297017955856424994 x^6 + 0.28632697921683422878 x^7 + 0.0076947084122278529318 x^9 + 1.2737765513849138734 x^4 + 0.069696699155259573138 x^8 + 0.026385901344907706743)$$

```
> plot(ChebPadeApp-f0(x),x=-1..1);
```



Maple commands introduced in this lesson:

quotients option for **cfrac**
periodic option for **cfrac**
chebpade(..., ..., [m,n]) in **numapprox** package