

Lesson 25: Solving equations using series

[> restart;

Example 2: A series for an implicit function

Find the Taylor series for $y(x)$ about $x=0$ (up to the x^6 term), if $y=y(x)$ satisfies the equation $(1+x)e^y - y^2 e^x = 1 + x^2 y$ with $y(0) = 0$.

Last time we saw that a version of Newton's method can be used.

```
> eq := (1+x)*exp(y)-y^2*exp(x) = 1 + x^2*y;
      eq := (1+x) e^y - y^2 e^x = 1 + x^2 y (1.1)
```

```
> f := unapply(lhs(eq)-rhs(eq), (x,y));
      f := (x,y) -> (1+x) e^y - y^2 e^x - 1 - x^2 y
```

```
> newt := (y,n) -> convert(normal(taylor(y-f(x,y)/D[2](f)(x,y),
      x, n)), polynom);
      newt := (y, n) -> convert( normal( taylor( y - \frac{f(x,y)}{D_2(f)(x,y)}, x, n ) ), polynom )
```

```
> y1 := newt(0,2);
      y1 := -x
```

```
> y2 := newt(y1,4);
      y2 := -x + \frac{3}{2} x^2 - \frac{10}{3} x^3
```

If $f(x, y_k) = O(x^k)$ and $\frac{\partial}{\partial y} f(x, y_k)$ has a nonzero limit as $x \rightarrow 0$, then $f(x, y_{k+1}) = O(x^{2k})$ where

$$y_{k+1} = y_k - \frac{f(x, y_k)}{\frac{\partial}{\partial y} f(x, y_k)}.$$

In other words, once you get an approximation that works to a certain order $O(x^k)$, each application of Newton's method will at least double the order of approximation.

```
> y3 := newt(y2,8);
      y3 := -x + \frac{3}{2} x^2 - \frac{10}{3} x^3 + \frac{23}{3} x^4 - \frac{1097}{60} x^5 + \frac{8117}{180} x^6 - \frac{285673}{2520} x^7
```

Notice that the terms in x, x^2, x^3 are the same as in y_2 .

```
> y4 := newt(y3, 16);
      y4 := -x + \frac{3}{2} x^2 - \frac{10}{3} x^3 + \frac{23}{3} x^4 - \frac{1097}{60} x^5 + \frac{8117}{180} x^6 - \frac{285673}{2520} x^7 + \frac{242153}{840} x^8
      - \frac{1061687}{1440} x^9 + \frac{1141506817}{604800} x^{10} - \frac{48062135701}{9979200} x^{11} + \frac{974043196177}{79833600} x^{12}
      - \frac{95110784043697}{3113510400} x^{13} + \frac{546125185861933}{7264857600} x^{14} - \frac{16853821021600523}{93405312000} x^{15}
```

I'll switch to using floating-point (by sticking in an `evalf`), because some of these coefficients are starting to involve rational numbers with big numerators and denominators.

```
> y5 := newt(evalf(y4), 32);
```

$$y5 := -1. x + 1.500000000 x^2 - 3.333333333 x^3 + 7.666666663 x^4 - 18.28333333 x^5$$

$$+ 45.09444446 x^6 - 113.3623015 x^7 + 288.2773811 x^8 - 737.2826390 x^9$$

$$+ 1887.412065 x^{10} - 4816.231328 x^{11} + 12200.91789 x^{12} - 30547.76497 x^{13}$$

$$+ 75173.55676 x^{14} - 1.804374995 10^5 x^{15} + 4.174699534 10^5 x^{16}$$

$$- 9.118492738 10^5 x^{17} + 1.800051234 10^6 x^{18} - 2.838448116 10^6 x^{19}$$

$$+ 1.536521088 10^6 x^{20} + 1.426525873 10^7 x^{21} - 9.335475933 10^7 x^{22}$$

$$+ 4.138114467 10^8 x^{23} - 1.589561380 10^9 x^{24} + 5.655720661 10^9 x^{25}$$

$$- 1.916297530 10^{10} x^{26} + 6.270482445 10^{10} x^{27} - 1.997374001 10^{11} x^{28}$$

$$+ 6.223365828 10^{11} x^{29} - 1.902341622 10^{12} x^{30} + 5.715120029 10^{12} x^{31}$$

This polynomial might not be sorted in order of the exponents. We can use `sort` to fix this.

```
> sort(y5, x, ascending);
```

$$-1. x + 1.500000000 x^2 - 3.333333333 x^3 + 7.666666663 x^4 - 18.28333333 x^5$$

$$+ 45.09444446 x^6 - 113.3623015 x^7 + 288.2773811 x^8 - 737.2826390 x^9$$

$$+ 1887.412065 x^{10} - 4816.231328 x^{11} + 12200.91789 x^{12} - 30547.76497 x^{13}$$

$$+ 75173.55676 x^{14} - 1.804374995 10^5 x^{15} + 4.174699534 10^5 x^{16}$$

$$- 9.118492738 10^5 x^{17} + 1.800051234 10^6 x^{18} - 2.838448116 10^6 x^{19}$$

$$+ 1.536521088 10^6 x^{20} + 1.426525873 10^7 x^{21} - 9.335475933 10^7 x^{22}$$

$$+ 4.138114467 10^8 x^{23} - 1.589561380 10^9 x^{24} + 5.655720661 10^9 x^{25}$$

$$- 1.916297530 10^{10} x^{26} + 6.270482445 10^{10} x^{27} - 1.997374001 10^{11} x^{28}$$

$$+ 6.223365828 10^{11} x^{29} - 1.902341622 10^{12} x^{30} + 5.715120029 10^{12} x^{31}$$

(1.2)

```
> y6 := sort(newt(evalf(y5), 64), x, ascending);
```

$$y6 := -1. x + 1.500000000 x^2 - 3.333333333 x^3 + 7.666666669 x^4 - 18.283333332 x^5$$

$$+ 45.09444442 x^6 - 113.3623018 x^7 + 288.2773810 x^8 - 737.2826395 x^9$$

$$+ 1887.412065 x^{10} - 4816.231337 x^{11} + 12200.91784 x^{12} - 30547.76496 x^{13}$$

$$+ 75173.55700 x^{14} - 1.804375001 10^5 x^{15} + 4.174699567 10^5 x^{16}$$

$$- 9.118492796 10^5 x^{17} + 1.800051253 10^6 x^{18} - 2.838448165 10^6 x^{19}$$

$$+ 1.536521332 10^6 x^{20} + 1.426525847 10^7 x^{21} - 9.335475828 10^7 x^{22}$$

$$+ 4.138114435 10^8 x^{23} - 1.589561372 10^9 x^{24} + 5.655720642 10^9 x^{25}$$

$$- 1.916297523 10^{10} x^{26} + 6.270482426 10^{10} x^{27} - 1.997374004 10^{11} x^{28}$$

$$+ 6.223365837 10^{11} x^{29} - 1.902341633 10^{12} x^{30} + 5.715120033 10^{12} x^{31}$$

$$- 1.689062000 10^{13} x^{32} + 4.912063172 10^{13} x^{33} - 1.405151490 10^{14} x^{34}$$

$$+ 3.949808232 10^{14} x^{35} - 1.088982176 10^{15} x^{36} + 2.936073135 10^{15} x^{37}$$

$$- 7.705055139 10^{15} x^{38} + 1.953187460 10^{16} x^{39} - 4.720377022 10^{16} x^{40}$$

$$+ 1.060602590 10^{17} x^{41} - 2.090715181 10^{17} x^{42} + 2.975312966 10^{17} x^{43}$$

(1.3)

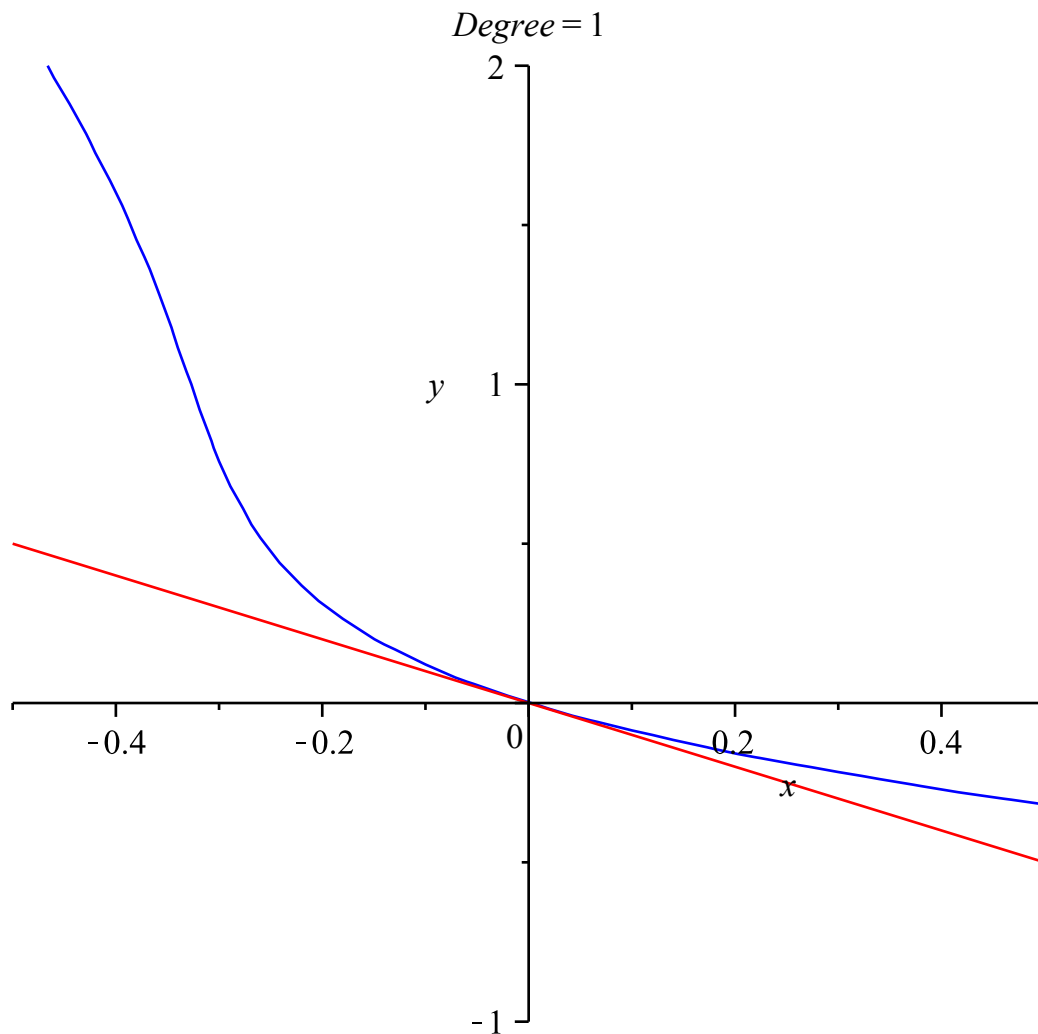
$$\begin{aligned}
&+ 8.86495164 \cdot 10^{16} x^{44} - 3.142658004 \cdot 10^{18} x^{45} + 1.773084316 \cdot 10^{19} x^{46} \\
&- 7.675919233 \cdot 10^{19} x^{47} + 2.959398805 \cdot 10^{20} x^{48} - 1.066902120 \cdot 10^{21} x^{49} \\
&+ 3.676706220 \cdot 10^{21} x^{50} - 1.225458658 \cdot 10^{22} x^{51} + 3.977643050 \cdot 10^{22} x^{52} \\
&- 1.262644071 \cdot 10^{23} x^{53} + 3.930219312 \cdot 10^{23} x^{54} - 1.201512023 \cdot 10^{24} x^{55} \\
&+ 3.610529448 \cdot 10^{24} x^{56} - 1.066640132 \cdot 10^{25} x^{57} + 3.096497034 \cdot 10^{25} x^{58} \\
&- 8.823296399 \cdot 10^{25} x^{59} + 2.462763065 \cdot 10^{26} x^{60} - 6.711763781 \cdot 10^{26} x^{61} \\
&+ 1.776763274 \cdot 10^{27} x^{62} - 4.530176727 \cdot 10^{27} x^{63}
\end{aligned}$$

Here's an animation with polynomials of degrees up to 63, showing how well this converges to a solution.

```

> with(plots):
P0:= implicitplot(f(x,y),x=-0.5 .. 0.5, y = -1 .. 2,colour=
blue):
> for j from 1 to 63 do
    frame[j]:= display([P0,plot(convert(taylor(y6,x,j+1),
polynom),x=-0.5..0.5)],title=('Degree'=j),view=[-0.5..0.5,-1.
.2])
end do:
display([seq(frame[j],j=1..63)],insequence=true);

```



What's the radius of convergence? From the pictures, I'd guess it's a bit more than 0.3.

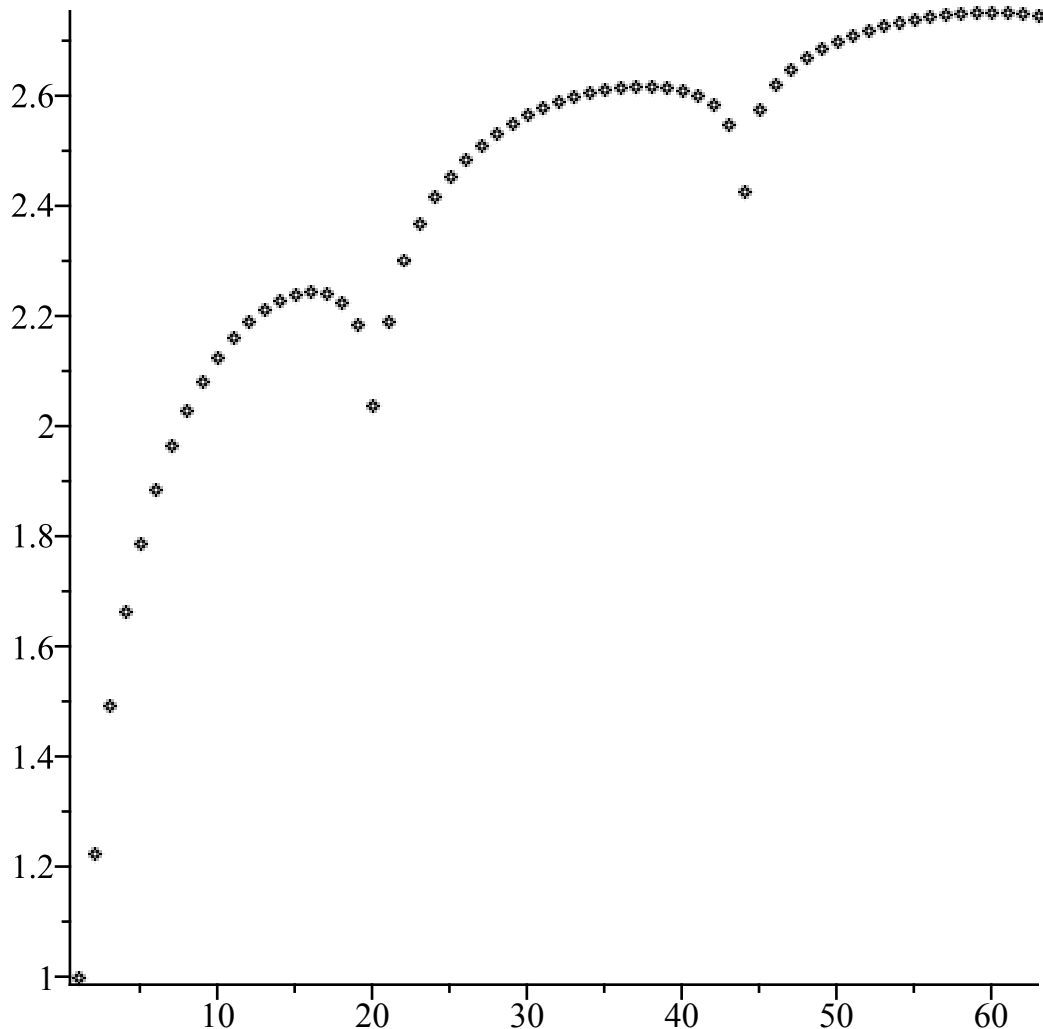
The theoretical result is that $\frac{1}{R} = \limsup_{n \rightarrow \infty} \left(|a_n|^{\frac{1}{n}} \right)$.

That means (in the case where R is finite and nonzero) that for every $\epsilon > 0$, $|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon$ when n is sufficiently large, but there exist arbitrarily large n with $\frac{1}{R} - \epsilon < |a_n|^{\frac{1}{n}}$.

```
> L := [seq([n, evalf(abs(coeff(y6, x, n))^(1/n))], n=1..63)];
L := [[1, 1.], [2, 1.224744871], [3, 1.493801582], [4, 1.663993576], [5, 1.788179350],
[6, 1.886631900], [7, 1.965601714], [8, 2.029907840], [9, 2.082696566], [10,
2.126114614], [11, 2.161680855], [12, 2.190446794], [13, 2.213086651], [14,
2.229925554], [15, 2.240896441], [16, 2.245376002], [17, 2.241732093], [18,
2.225952159], [19, 2.185927936], [20, 2.038576107], [21, 2.191189727], [22,
2.302920416], [23, 2.369428385], [24, 2.417611730], [25, 2.455271473], [26,
2.485875326], [27, 2.511308319], [28, 2.532723823], [29, 2.550882936], [30,
2.566312960], [31, 2.579389615], [32, 2.590382663], [33, 2.599481830], [34,
```

2.606810757], [35, 2.612432205], [36, 2.616345063], [37, 2.618470909], [38, 2.618623410], [39, 2.616443631], [40, 2.611255797], [41, 2.601700992], [42, 2.584568061], [43, 2.548949460], [44, 2.427585099], [45, 2.576623997], [46, 2.620901135], [47, 2.649036125], [48, 2.669828754], [49, 2.686243636], [50, 2.699660664], [51, 2.710840063], [52, 2.720246052], [53, 2.728181285], [54, 2.734850837], [55, 2.740395455], [56, 2.744909635], [57, 2.748451222], [58, 2.751045480], [59, 2.752684614], [60, 2.753322247], [61, 2.752860521], [62, 2.751123929], [63, 2.747805286]]

```
> pointplot(L);
```

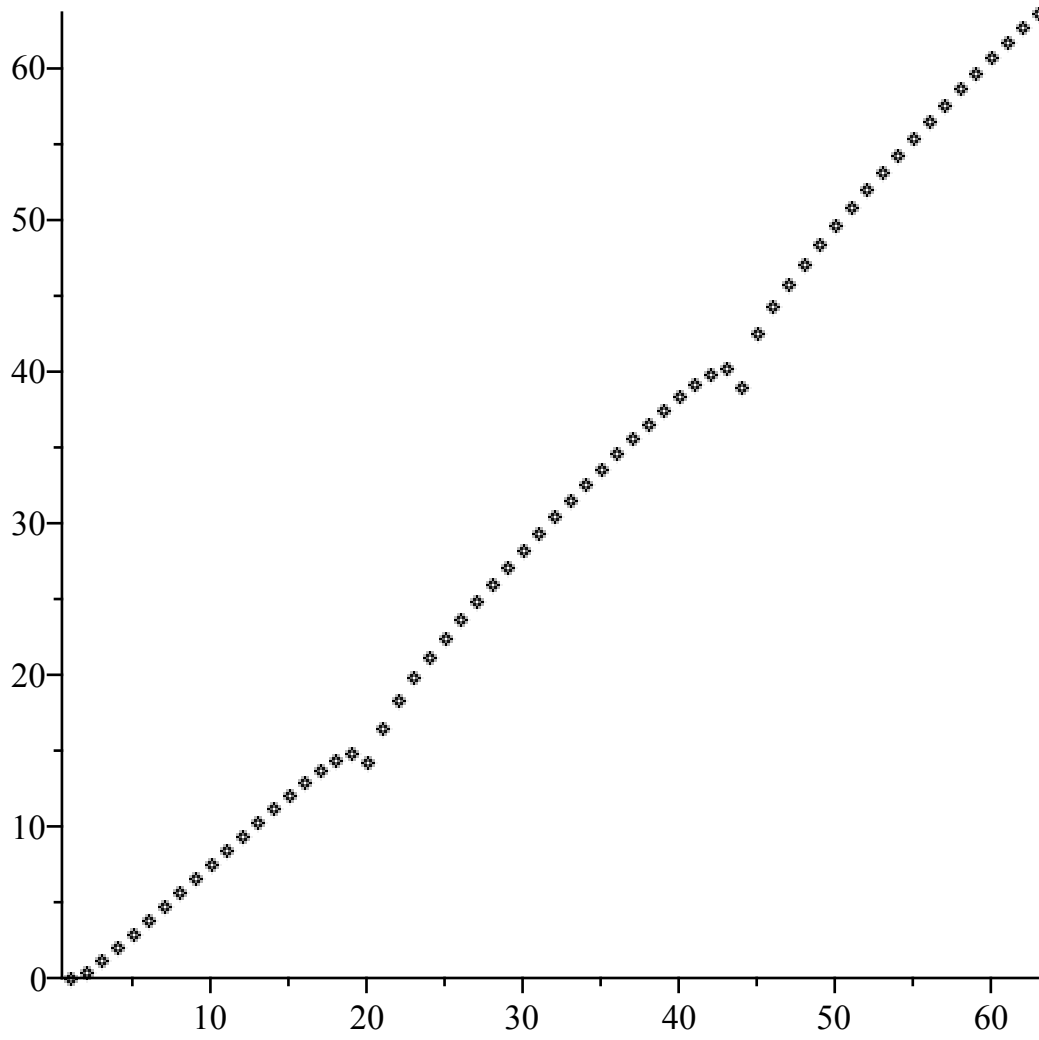


It looks plausible (with a little imagination) that the lim sup is around 3, which would correspond to a radius of about 0.3.

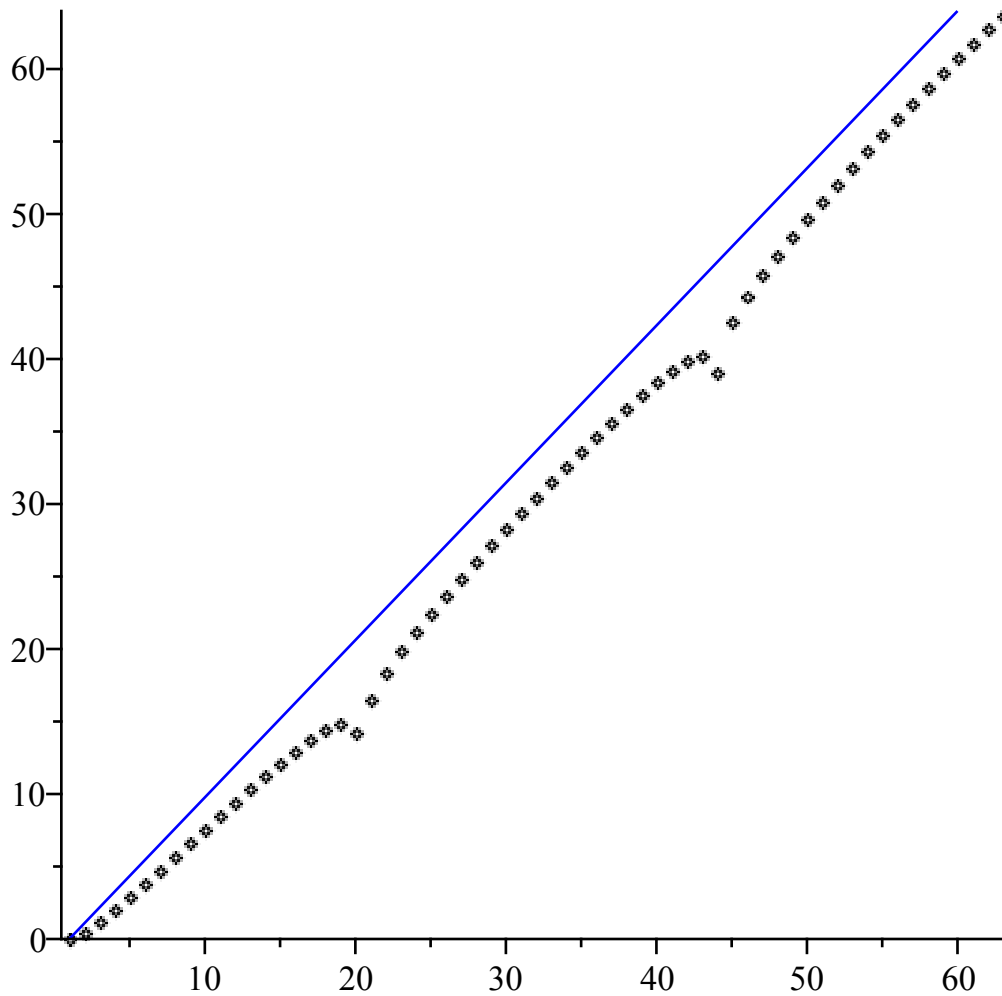
Here's a different (possibly better) way to do it: plot $\ln(|a_n|)$. The idea here is that if $R > r$, then $|a_n| < c r^{-n}$ for some constant c , so $\ln(|a_n|) < \ln(c) - n \ln(r)$. Thus in this plot, every point should be below a straight line with slope $-\ln(r)$. The radius of convergence R is e^{-m} where m is the minimum slope such that all points are below a line of slope m .

```
> L2 := [seq([n, evalf(ln(abs(coeff(y6,x,n))))], n=1..63)]:
```

```
> pointplot(L2);P1:= %:
```



```
> display([P1, plot([L2[1],[60,64]],colour=blue)]);
```



```
> slope:= (64-L2[1,2])/(60-L2[1,1]);
              slope := 1.084745763
```

(1.4)

```
> R:= exp(-slope);
              R := 0.3379877040
```

(1.5)

By the way, here's another way to get the series.

```
> solve(eq,y);
              RootOf(-e-Z - e-Zx + 1 + _Z2ex + x2_Z)
```

(1.6)

```
> taylor(%,x,10);
-x +  $\frac{3}{2}x^2 - \frac{10}{3}x^3 + \frac{23}{3}x^4 - \frac{1097}{60}x^5 + \frac{8117}{180}x^6 - \frac{285673}{2520}x^7 + \frac{242153}{840}x^8$ 
-  $\frac{1061687}{1440}x^9 + O(x^{10})$ 
```

(1.7)

```
> taylor(%% - y4, x, 16);
              O(x16)
```

(1.8)

There's something slightly fishy about this:

where did I tell Maple that I wanted the solution with $y(0) = 0$?

I didn't. Now it just happens that the only real solution at $x = 0$ is $y = 0$. But for equations that don't

have that property, you might not know which solution you'll get by this method. For example, the next example has two solutions at $x=0$: $y=1$ and $y=-1$.

$$\begin{aligned} > \text{eq2} := y^2 + x \cdot \exp(y) = 1; \\ & \qquad \qquad \qquad \text{eq2} := y^2 + e^y x = 1 \end{aligned} \tag{1.9}$$

$$\begin{aligned} > \text{solve}(\text{eq2}, y); \\ & \qquad \qquad \qquad \text{RootOf}(e^{-Z} x + _Z^2 - 1) \end{aligned} \tag{1.10}$$

$$\begin{aligned} > \text{taylor}(\%, x, 10); \\ -1 + \frac{1}{2} e^{-1} x + \frac{3}{8} e^{-2} x^2 + \frac{7}{16} e^{-3} x^3 + \frac{235}{384} e^{-4} x^4 + \frac{121}{128} e^{-5} x^5 + \frac{7959}{5120} e^{-6} x^6 \\ + \frac{245953}{92160} e^{-7} x^7 + \frac{5422687}{1146880} e^{-8} x^8 + \frac{3936241}{458752} e^{-9} x^9 + O(x^{10}) \end{aligned} \tag{1.11}$$

I get the solution with $y(0) = -1$, but I wouldn't have known this ahead of time. If I want the solution with $y(0) = 1$, I could put an extra argument on the RootOf that says what the solution should be at $x=0$.

$$\begin{aligned} > \text{taylor}(\text{RootOf}(\exp(_Z) * x + _Z^2 - 1, 1), x, 10); \\ 1 - \frac{1}{2} e x + \frac{1}{8} e^2 x^2 - \frac{1}{16} e^3 x^3 + \frac{13}{384} e^4 x^4 - \frac{1}{48} e^5 x^5 + \frac{69}{5120} e^6 x^6 - \frac{841}{92160} e^7 x^7 \\ + \frac{65689}{10321920} e^8 x^8 - \frac{10427}{2293760} e^9 x^9 + O(x^{10}) \end{aligned} \tag{1.12}$$

From series to function

So far we've had a function and wanted to know its Taylor series. Now suppose you know the series but you want to identify the function. Maple might be able to do it with **sum**, if you know a formula for the coefficients.

$$\begin{aligned} > \text{sum}(k/(k+1) * x^k, k=0..infinity); \\ \frac{1}{2} x \left(-\frac{2}{x(x-1)} + \frac{2 \ln(1-x)}{x^2} \right) \end{aligned}$$

It's pretty good at doing sums.

$$\begin{aligned} > \text{sum}(((1+k)!)^2 / (1+2*k)! * x^k, k=0..infinity); \\ \frac{3}{4 \left(\frac{1}{4} x - 1 \right)^2} + \frac{1}{2} \frac{\left(-1 - \frac{1}{2} x \right) \sqrt{1 - \frac{1}{4} x} \arcsin\left(\frac{1}{2} \sqrt{x} \right)}{\left(\frac{1}{4} x - 1 \right)^3 \sqrt{x}} \end{aligned}$$

$$\begin{aligned} > \text{sum}((2*k+1)^2 / (k!)^2 * x^k, k=0..infinity); \\ \text{BesselI}(0, 2 \sqrt{x}) + 4 \sqrt{x} \text{BesselI}(1, 2 \sqrt{x}) + 4 x \text{BesselI}(0, 2 \sqrt{x}) \end{aligned}$$

Of course, sometimes there's no "closed form" formula.

$$\begin{aligned} > \text{sum}(x^k / (1+2^k), k=0..infinity); \\ \sum_{k=0}^{\infty} \frac{x^k}{1+2^k} \end{aligned}$$

But suppose you only know a finite number of terms of the series. Is there any hope? Theoretically, no: the series could continue in all sorts of ways, e.g. the coefficients might all be 0 from this point

on. But Maple might be able to "guess" how it continues. The appropriate function is **guessgf** in the **gfun** package. Here's a list of numbers.

```
> L := [1/2,1/4,1/6,1/8,1/10];
```

$$L := \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10} \right]$$

What's a likely function whose Maclaurin series is $L_1 + L_2 x + L_3 x^2 + \dots$?

```
> with(gfun):
```

```
> guessgf(L,x,[ogf]);
```

$$\left[-\frac{1}{2} \frac{\ln(1-x)}{x}, ogf \right]$$

```
> taylor(%[1],x,20);
```

$$\begin{aligned} & \frac{1}{2} + \frac{1}{4} x + \frac{1}{6} x^2 + \frac{1}{8} x^3 + \frac{1}{10} x^4 + \frac{1}{12} x^5 + \frac{1}{14} x^6 + \frac{1}{16} x^7 + \frac{1}{18} x^8 + \frac{1}{20} x^9 \\ & + \frac{1}{22} x^{10} + \frac{1}{24} x^{11} + \frac{1}{26} x^{12} + \frac{1}{28} x^{13} + \frac{1}{30} x^{14} + \frac{1}{32} x^{15} + \frac{1}{34} x^{16} + \frac{1}{36} x^{17} \\ & + \frac{1}{38} x^{18} + O(x^{19}) \end{aligned}$$

```
> convert(%[1],FormalPowerSeries,x);
```

$$\sum_{k=0}^{\infty} \frac{x^k}{2k+2}$$

(2.1)

That was easy. Here's one that's not quite so obvious.

```
> guessgf([1,2,4,7,11,16,22],x,[ogf]);
```

$$\left[\frac{-1+x-x^2}{(x-1)^3}, ogf \right]$$

```
> taylor(%[1],x,8);
```

$$1 + 2x + 4x^2 + 7x^3 + 11x^4 + 16x^5 + 22x^6 + 29x^7 + O(x^8)$$

```
> convert(%[1],FormalPowerSeries,x);
```

$$\sum_{k=0}^{\infty} \left(1 + \frac{1}{2} k + \frac{1}{2} k^2 \right) x^k$$

(2.2)

The **ogf** stands for "ordinary generating function". A function $f(x)$ is the ordinary generating function of the sequence c_0, c_1, c_2, \dots if that sequence is the sequence of Maclaurin series

coefficients of $f(x)$, i.e. $f(x) = \sum_{k=0}^{\infty} c_k x^k$

There's also **egf** or "exponential generating function", for a function whose coefficients are $\frac{L_1}{0!}$,

$\frac{L_2}{1!}, \frac{L_3}{2!}, \dots$. That's not as useful for us here.

Also in the package is **listtoalgeq**, which would find a polynomial equation in x and y satisfied when y is the series with coefficients given by the list. It wouldn't work for our first implicit example, because the equation there involved exponentials. But try this one:

$x^2y - xy^3 + y - 1 = 0$ with $y(0) = 1$.

```
> f := (x,y) -> x^2*y - x*y^3 + y - 1;  
f := (x, y) → x2y - xy3 + y - 1
```

Note that $f(0, 1) = 0$. I'll use the Newton's method trick to find the Taylor series of the solution $y(x)$ about $x = 0$, then see if **listtoalgeq** will find the equation that $y(x)$ satisfies.

```
> newt := (y,n) -> convert(normal(taylor(y-f(x,y)/D[2](f)(x,y),  
x=0, n)),polynom);
```

```
newt := (y, n) → convert( normal( taylor( y -  $\frac{f(x,y)}{D_2(f)(x,y)}$ , x=0, n ) ), polynom )
```

```
> Y[0] := 1;  
for j from 1 to 5 do  
  Y[j] := newt(Y[j-1],2^j)  
end do;
```

$$Y_0 := 1$$

$$Y_1 := 1 + x$$

$$Y_2 := 1 + x + 2x^2 + 8x^3$$

$$Y_3 := 1 + x + 2x^2 + 8x^3 + 35x^4 + 163x^5 + 796x^6 + 4024x^7$$

$$Y_4 := 1 + x + 2x^2 + 8x^3 + 35x^4 + 163x^5 + 796x^6 + 4024x^7 + 20885x^8 + 110654x^9 + 596064x^{10} + 3254752x^{11} + 17974893x^{12} + 100227022x^{13} + 563482140x^{14} + 3190633232x^{15}$$

$$Y_5 := 1 + x + 2x^2 + 8x^3 + 35x^4 + 163x^5 + 796x^6 + 4024x^7 + 20885x^8 + 110654x^9 + 596064x^{10} + 3254752x^{11} + 17974893x^{12} + 100227022x^{13} + 563482140x^{14} + 3190633232x^{15} + 18179765509x^{16} + 104158703503x^{17} + 599698459613x^{18} + 3467978715612x^{19} + 20134256546896x^{20} + 117313279477959x^{21} + 685756774642494x^{22} + 4020515276730588x^{23} + 23636036336651811x^{24} + 139301059260764048x^{25} + 822881759633309667x^{26} + 4871350637075703196x^{27} + 28895082181969536230x^{28} + 171712367070082813220x^{29} + 1022183276503900838428x^{30} + 6094767743827565180092x^{31}$$

Can Maple take the list of coefficients and get the equation?

```
> L := [seq(coeff(Y[5],x,j),j=0..31)];
```

```
L := [1, 1, 2, 8, 35, 163, 796, 4024, 20885, 110654, 596064, 3254752, 17974893,  
100227022, 563482140, 3190633232, 18179765509, 104158703503, 599698459613,  
3467978715612, 20134256546896, 117313279477959, 685756774642494,  
4020515276730588, 23636036336651811, 139301059260764048,  
822881759633309667, 4871350637075703196, 28895082181969536230,  
171712367070082813220, 1022183276503900838428, 6094767743827565180092]
```

```
> listtoalgeq(L,y(x));
```

$$[1 + (-1 - x^2)y(x) + xy(x)^3, ogf]$$

The first element here is equivalent to our $f(x, y(x))$ (well, actually it's $-f(x, y(x))$).

```
> %[1] + f(x,y(x));
          (-1 - x^2) y(x) + x^2 y(x) + y(x)
> expand(%);
          0
(2.3)
```

```
> convert(RootOf(f(x,y),y),FormalPowerSeries,x);
          RootOf(1 + x _Z^3 + (-1 - x^2) _Z)
(2.4)
```

By the way, there's another interesting method of identifying a sequence of integers: the Encyclopedia of Integer Sequences. For example, what about the sequence 1, 1, 3, 15, 108, 1032, 12388, ...?

Look it up at <http://www.research.att.com/~njas/sequences/index.html>

A functional equation

The Encyclopedia of Integer Sequences says the ordinary generating function of sequence A090351 (i.e. the function whose Maclaurin series is that sequence) satisfies the equation

$$A(x)^3 = \frac{A\left(\frac{x}{1-x}\right)^2}{1-x}$$

That's called a functional equation: an equation involving values of an unknown function at different points. Can we use Maple to solve it, recovering the sequence?

```
> eq:= A(x)^3 = A(x/(1-x))^2/(1-x);
          eq := A(x)^3 = \frac{A\left(\frac{x}{1-x}\right)^2}{1-x}
```

Note that for $x=0$ we get

```
> eval(eq, x=0);
          A(0)^3 = A(0)^2
```

so $A(0) = 0$ or 1 . The one we want is 1 .

```
> Aseries := unapply(1 + add(a[j]*x^j, j=1..20), x);
Aseries := x → 1 + a1x + a2x2 + a3x3 + a4x4 + a5x5 + a6x6 + a7x7 + a8x8 + a9x9
          + a10x10 + a11x11 + a12x12 + a13x13 + a14x14 + a15x15 + a16x16 + a17x17 + a18x18
          + a19x19 + a20x20
```

The fact that $\frac{x}{1-x} = 0$ when $x=0$ makes it possible to substitute this series in to the equation.

```
> eval(eq, A = Aseries);
(1 + a1x + a2x2 + a3x3 + a4x4 + a5x5 + a6x6 + a7x7 + a8x8 + a9x9 + a10x10 + a11x11
  + a12x12 + a13x13 + a14x14 + a15x15 + a16x16 + a17x17 + a18x18 + a19x19
  + a20x20)3 = \frac{1}{1-x} \left( 1 + \frac{a_1 x}{1-x} + \frac{a_2 x^2}{(1-x)^2} + \frac{a_3 x^3}{(1-x)^3} + \frac{a_4 x^4}{(1-x)^4} \right)
```

$$\begin{aligned}
& + \frac{a_5 x^5}{(1-x)^5} + \frac{a_6 x^6}{(1-x)^6} + \frac{a_7 x^7}{(1-x)^7} + \frac{a_8 x^8}{(1-x)^8} + \frac{a_9 x^9}{(1-x)^9} + \frac{a_{10} x^{10}}{(1-x)^{10}} \\
& + \frac{a_{11} x^{11}}{(1-x)^{11}} + \frac{a_{12} x^{12}}{(1-x)^{12}} + \frac{a_{13} x^{13}}{(1-x)^{13}} + \frac{a_{14} x^{14}}{(1-x)^{14}} + \frac{a_{15} x^{15}}{(1-x)^{15}} + \frac{a_{16} x^{16}}{(1-x)^{16}} \\
& + \frac{a_{17} x^{17}}{(1-x)^{17}} + \frac{a_{18} x^{18}}{(1-x)^{18}} + \frac{a_{19} x^{19}}{(1-x)^{19}} + \frac{a_{20} x^{20}}{(1-x)^{20}} \Big)^2
\end{aligned}$$

In the PDF version of this lesson, I'll use a colon on the next two commands, so those who print them out don't waste a lot of paper and ink.

```

> taylor(lhs(%) - rhs(%) , x, 21):
> eqs := {seq(coeff(%, x, j), j=0..20)}:
> s := solve(eqs);
S := {a1 = 1, a2 = 3, a3 = 15, a4 = 108, a5 = 1032, a6 = 12388, a7 = 179572, a8 = 3052986, a9
= 59555338, a10 = 1310677726, a11 = 32114051862, a12 = 866766965308, a13
= 25547102523604, a14 = 816335926158372, a15 = 28107705687291892, a16
= 1037367351120788551, a17 = 40852168787823027351, a18
= 1709792654612819858341, a19 = 75786181910268208068217, a20
= 3546463856783571869500968}

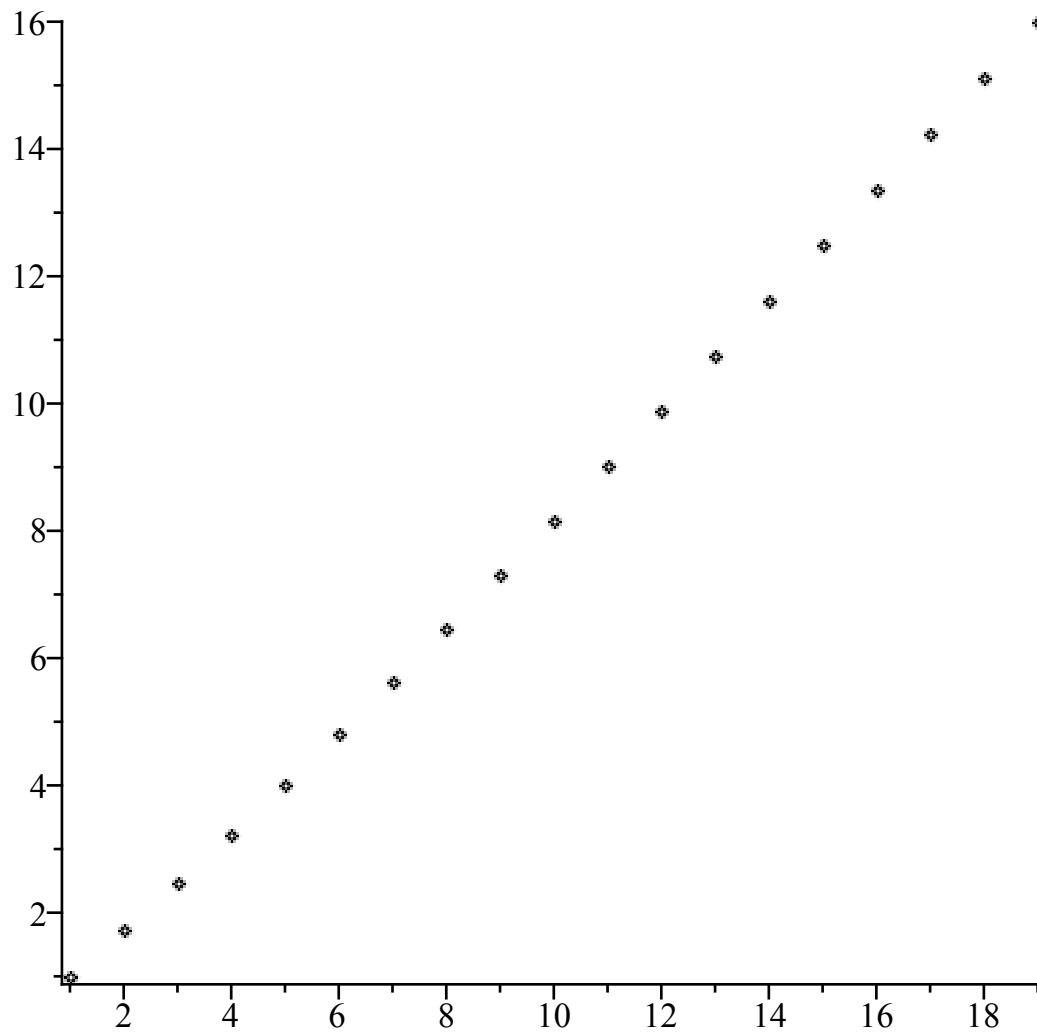
```

These numbers get large rather quickly. It's not at all obvious that the radius of convergence of the series would be positive.

```

> [seq([n, evalf(eval(a[n], s)^(1/n))], n=1..19)];
[[1, 1.], [2, 1.732050808], [3, 2.466212074], [4, 3.223709795], [5, 4.006230560], [6,
4.810241337], [7, 5.631259190], [8, 6.465328582], [9, 7.309372199], [10,
8.161116771], [11, 9.018910982], [12, 9.881552808], [13, 10.74815630], [14,
11.61805675], [15, 12.49074550], [16, 13.36582547], [17, 14.24298081], [18,
15.12195597], [19, 16.00254101]]
> with(plots):
pointplot(%);

```



It looks like $\lim_{n \rightarrow \infty} a_n^{\left(\frac{1}{n}\right)} = \infty$, so the radius of convergence is 0.

Maple objects introduced in this lesson

- gfun** package
- guessgf** (in **gfun**)
- listtoalgeq** (in **gfun**)