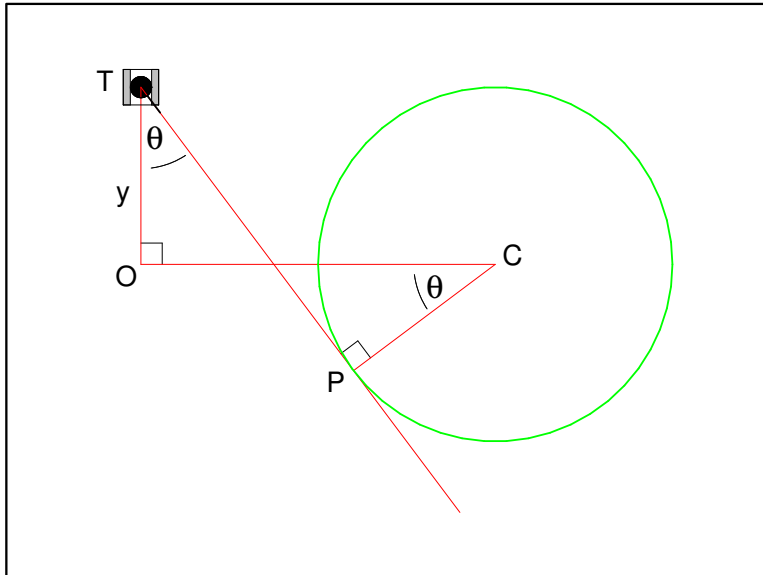


A Tank Problem

A tank is travelling down the y axis at 10 m/s. There is a hedge in the form of a circle of radius 50 m centred at the point $(100, 0)$. The soldiers in the tank suspect an enemy is hiding behind the hedge, so as they move they rotate the gun turret so that the gun is always pointing along a line tangent to the hedge. What is the fastest rate the turret must be able to rotate in order for this to work? Where is the tank when the turret is rotating at its maximum rate?

Let y be the position of the tank on the y axis, and θ the angle of rotation of the turret. Here is a diagram, with the tank at T, the hedge centred at C, and the line of the gun tangent to the hedge at P.



Note that θ is the angle OCP as well as OTP. Thus $P = (100 - 50 \cos(\theta), -50 \sin(\theta))$. The slope of PC is TP is $\tan(\theta)$, so the slope of TP is $-\cot(\theta)$. Thus

$$\frac{y + 50 \sin(\theta)}{100 - 50 \cos(\theta)} = \cot(\theta)$$

We can simplify this somewhat: write $\cot(\theta) = \cos(\theta)/\sin(\theta)$ and multiply both sides by $\sin(\theta)(100 - 50 \cos(\theta))$ to get

$$y \sin(\theta) + 50 \sin^2(\theta) = 100 \cos(\theta) - 50 \cos^2(\theta)$$

Put the $50 \sin^2(\theta)$ on the other side, and use the identity $\sin^2(\theta) + \cos^2(\theta) = 1$:

$$y \sin(\theta) = 100 \cos(\theta) - 50$$

I'll come back later to the question of how to solve this for θ as a function of y , but for now we can use implicit differentiation on this equation.

$$\frac{dy}{dt} \sin(\theta) + y \cos(\theta) \frac{d\theta}{dt} = -100 \sin(\theta) \frac{d\theta}{dt}$$

Using $\frac{dy}{dt} = -10$, we can solve for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{10 \sin(\theta)}{y \cos(\theta) + 100 \sin(\theta)}$$

We can use $y = (100 \cos(\theta) - 50) / \sin(\theta)$ to get this as a function of θ alone:

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{10 \sin(\theta)}{(100 \cos(\theta) - 50) \frac{\cos(\theta)}{\sin(\theta)} + 100 \sin(\theta)} \\ &= \frac{\sin^2(\theta)}{10 \cos^2(\theta) + 10 \sin^2(\theta) - 5 \cos(\theta)} \\ &= \frac{\sin^2(\theta)}{10 - 5 \cos(\theta)} \end{aligned}$$

Note that this is always positive: the turret will always be rotating counterclockwise. Writing the right side of the last equation as $F(\theta)$, we want to find the maximum of $F(\theta)$ for $0 \leq \theta \leq \pi$. Looking for critical points, we have

$$\begin{aligned} F'(\theta) &= \frac{(10 - 5 \cos(\theta))(2 \sin(\theta) \cos(\theta)) - 5 \sin^3(\theta)}{(10 - 5 \cos(\theta))^2} \\ &= \frac{\sin(\theta)}{(10 - 5 \cos(\theta))^2} (20 \cos(\theta) - 10 \cos^2(\theta) - 5 \sin^2(\theta)) \\ &= \frac{\sin(\theta)}{(10 - 5 \cos(\theta))^2} (20 \cos(\theta) - 5 \cos^2(\theta) - 5) \end{aligned}$$

So for a critical point in the interval, we must have $4 \cos(\theta) - \cos^2(\theta) - 1 = 0$. Think of this as a quadratic equation $4x - x^2 - 1 = 0$ where $x = \cos(\theta)$. The solutions of the quadratic are $x = 2 \pm \sqrt{3}$, but since $\cos(\theta)$ must be between -1 and 1 , the only allowed solution is $\cos(\theta) = 2 - \sqrt{3}$. This corresponds to

$$\begin{aligned} F(\theta) &= \frac{\sin^2(\theta)}{10 - 5 \cos(\theta)} = \frac{1 - \cos^2(\theta)}{10 - 5 \cos(\theta)} \\ &= \frac{1 - (2 - \sqrt{3})^2}{5\sqrt{3}} = \frac{4 - 2\sqrt{3}}{5} \end{aligned}$$

At the endpoints we have $F(0) = F(\pi) = 0$, while this value is positive (approximately .10718 according to the calculator), so this value must be the maximum: the turret must be able to rotate at up to .10718 radians per second.

This maximum occurred when $\cos(\theta) = 2 - \sqrt{3}$, so (since $0 < \theta < \pi$ is the correct interval for the arccos function) $\theta = \arccos(2 - \sqrt{3}) \approx 1.2995$ radians, corresponding to

$$\begin{aligned} y &= \frac{100 \cos(\theta) - 50}{\sin(\theta)} = \frac{150 - 100\sqrt{3}}{\sqrt{1 - (2 - \sqrt{3})^2}} \\ &= \frac{150 - 100\sqrt{3}}{\sqrt{4\sqrt{3} - 6}} \end{aligned}$$

or approximately -24.086 metres.

θ as a function of y

We have answered the question, but it would be nice to be able to write θ as a function of y . So we go back to the equation

$$y \sin(\theta) = 100 \cos(\theta) - 50$$

If $\cos(\theta) = x$, $\sin(\theta) = \sqrt{1 - x^2}$ (remembering again that $0 < \theta < \pi$), so this becomes

$$y\sqrt{1 - x^2} = 100x - 50$$

Square both sides to get rid of the square root:

$$y^2(1 - x^2) = 10000x^2 - 10000x + 2500$$

Consider this as a quadratic equation in x :

$$(y^2 + 10000)x^2 - 10000x + 2500 - y^2 = 0$$

This has solutions

$$x = \frac{5000 \pm \sqrt{y^4 + 7500y^2}}{10000 + y^2} = \frac{5000 \pm y\sqrt{y^2 + 7500}}{10000 + y^2}$$

Which is it, + or -? Well, $\frac{5000 + y\sqrt{y^2 + 7500}}{10000 + y^2} \rightarrow +1$ as $y \rightarrow +\infty$ and -1 as $y \rightarrow -\infty$, while the limits for $\frac{5000 - y\sqrt{y^2 + 7500}}{10000 + y^2}$ would be -1 and 1 respectively. We want $x = \cos(\theta) \rightarrow +1$ as $y \rightarrow +\infty$ and -1 as $y \rightarrow -\infty$, so we want the +. Thus (again since θ is in the correct interval)

$$\theta = \arccos(x) = \arccos\left(\frac{5000 + y\sqrt{y^2 + 7500}}{10000 + y^2}\right)$$

Here is a plot of θ as a function of y :

