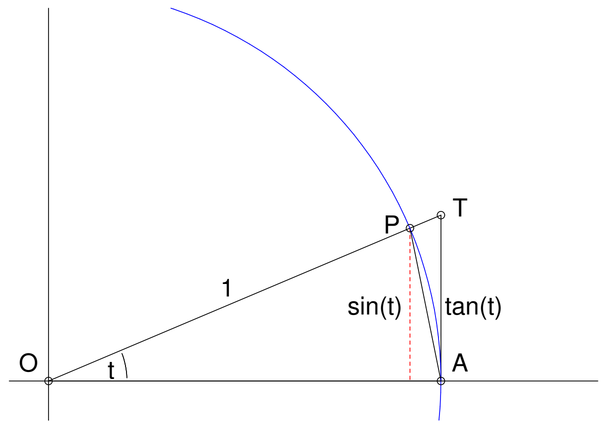
Proof of a limit

Theorem. $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$

Suppose $0 < t < \pi/2$. Consider the diagram below, where P is the point $(\cos(t), \sin(t))$ on the circle of radius 1 centred at the origin O, and $A = (1, 0)$. The straight line OP intersects the line $x = 1$ at $T = (1, \tan(t))$.



The sector of the circle from OA to OP constitutes a fraction $t/(2\pi)$ of the circle. The whole circle has area π, so this sector has area $t/2$. It contains triangle OAP and is contained in triangle OAT, so

$$\text{Area}(OAP) < t/2 < \text{Area}(OAT)$$

Triangle OAP has base 1 and height $\sin(t)$, so its area is $\sin(t)/2$. Triangle OAT has base 1 and height $\tan(t)$, so its area is $\tan(t)/2$. Thus we have

$$\frac{\sin(t)}{2} < \frac{t}{2} < \frac{\tan(t)}{2} = \frac{\sin(t)}{2\cos(t)}$$

Taking reciprocals of positive numbers reverses inequalities, so

$$\frac{2}{\sin(t)} > \frac{2}{t} > \frac{2\cos(t)}{\sin(t)}$$

Multiplying by the positive number $\sin(t)/2$, we get

$$1 > \frac{\sin(t)}{t} > \cos(t)$$

This was for $0 < t < \pi/2$, but since both $\sin(t)/t$ and $\cos(t)$ are even functions, it's also true for $-\pi/2 < t < 0$. Now we know $\cos(t)$ is continuous, so as $t \to 0$ we have $\cos(t) \to \cos(0) = 1$. Since $\sin(t)/t$ is "squeezed" between 1 and a function that goes to 1, it must also go to 1 as $t \to 0$.