Estimates of the form
\[ \|f\|_p \leq A(N) \left( \sum_{j=1}^{N} \|f_j\|_p^p \right)^{1/p} \quad (p > 2) \]
and
\[ \|f\|_p \leq B(N) \left( \sum_{j=1}^{N} \|f_j\|_p^2 \right)^{1/2}, \]
where \( f = \sum_{j=1}^{N} f_j \), are ubiquitous in decoupling theory. In this lecture, we will discuss how to convert between such estimates using interpolation.

**Note.** In the following, if \( X \subseteq \mathbb{R}^n \), then \( L^p(X) \) will denote the vector space \( L^p(X, \Sigma, \mu) \), where \( \Sigma \) is the collection of Lebesgue-measurable subsets of \( X \) and \( \mu \) is Lebesgue measure. If \( I \subseteq \mathbb{N} \), then \( \ell^p(I) \) will denote the vector space \( L^p(I, \mathcal{P}(I), \mu) \), where \( \mu \) is counting measure. When the underlying set is clear from the context, we will write \( L^p \) (resp. \( \ell^p \)) for \( L^p(X) \) (resp. \( \ell^p(I) \)).

### The Riesz-Thorin theorem

Let us begin by recalling the Riesz-Thorin theorem.

**Theorem 1.1** (Riesz-Thorin). Let \( X \) and \( Y \) be \( \sigma \)-finite measure spaces, and \( p_0, p_1, q_0, q_1 \in [1, \infty] \). Suppose that \( T : L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(Y) + L^{q_1}(Y) \) is a linear operator with
\[ \|T\|_{L^{p_0}(X) \to L^{q_0}(Y)} \leq M_0 \quad \text{and} \quad \|T\|_{L^{p_1}(X) \to L^{q_1}(Y)} \leq M_1. \]
Then for all \( \theta \in (0,1) \), the operator \( T \) maps \( L^{p_\theta}(X) \) into \( L^{q_\theta}(Y) \) with
\[ \|T\|_{L^{p_\theta}(X) \to L^{q_\theta}(Y)} \leq M_0^{1-\theta} M_1^\theta, \]
where
\[ \frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}. \]


**Remark.** If \( T : L^p(X) \to L^q(Y) \) is a bounded linear operator, we say that \( T \) is of (strong) type \( (p,q) \). The Riesz-Thorin theorem implies that the set of all points \( \left( \frac{1}{p}, \frac{1}{q} \right) \) in the unit square such that \( T \) is of type \( (p,q) \) (sometimes called the Riesz diagram of \( T \)) is a convex set.
Among other results, the Riesz-Thorin theorem can be used to prove the Hausdorff-Young inequality and Young’s convolution inequality; see Bergh [1, Section 1.2]. As an illustration, we will use it to prove finite-dimensional norm inequalities, although these can be proven using more elementary methods (e.g., Hölder’s inequality).

**Example 1.2.** Let $X = Y = [N] := \{1, 2, \ldots, N\}$ and suppose that $1 \leq q \leq p \leq \infty$. Since $\|x\|_{q_I} = \sum_{j=1}^N |x_j|^q \leq N \|x\|_{q,\infty}$, we have $\|I\|_{\ell^q \to \ell^q} \leq N^{1/q}$ (which also holds when $q = \infty$). We also have $\|I\|_{\ell^p \to \ell^q} \leq 1$, so $\|I\|_{\ell^p \to \ell^q} \leq (N^{1/q})^{1-\theta} 1^\theta = (N^{1/q})^{1-q/p} = N^{1-1/p}$.

On the other hand, if $1 \leq p \leq q \leq \infty$, we begin by observing that $\|x\|_{q,\infty} = \max_{1 \leq j \leq N} |x_j|^p \leq \sum_{j=1}^N |x_j|^p = \|x\|_{p_I}$, which implies that $\|I\|_{\ell^p \to \ell^\infty} \leq 1$ (which also holds when $p = \infty$). We also have $\|I\|_{\ell^p \to \ell^p} \leq 1$, so $\|I\|_{\ell^p \to \ell^p} \leq 1$.

Together these inequalities imply that

$$\|x\|_{\ell^p} \leq \|x\|_{\ell^q} \leq N^{1/p-1/q} \|x\|_{\ell^q} \quad (\ast)$$

for all $x \in \mathbb{C}^N$ if $1 \leq p \leq q \leq \infty$.

**Interpolation spaces**

The idea of ‘interpolation’ between vector spaces can be greatly generalized. In this general setting, the language of category theory – briefly reviewed below – is used, although the underlying interpolation techniques make no use of the theory proper. In particular, for all intents and purposes, one may assume that “object” refers to a normed vector space (or a pair of such – see, for example, Definition 1.5) and “morphism” to a bounded linear operator.

A category $\mathcal{C}$ consists of a class of objects and a class of morphisms (also called arrows) between the objects such that if $f$ is a morphism from $A$ to $B$ and $g$ is a morphism from $B$ to $C$, there exists a morphism $g \circ f$ (also written $gf$) from $A$ to $C$. This binary operation of composing morphisms is required to be associative; and for every object $A$, there must exist a morphism $\text{id}_A$ (also written $1_A$) from $A$ to itself satisfying $\text{id}_A \circ f = f$ and $g \circ \text{id}_A = g$ for all morphisms $f$ and $g$ to and from $A$. If $f$ is a morphism from $A$ to $B$, we write $f : A \to B$.

A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ consists of a subclass of the objects of $\mathcal{C}$ and a subclass of the morphisms of $\mathcal{C}$ that constitute a category in and of themselves. If $\mathcal{S}$ contains all the morphisms that were originally in $\mathcal{C}$ between its own objects, it is called a full subcategory.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a (covariant) functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ is a map that associates each object $A$ in $\mathcal{C}$ with an object $F(A)$ in $\mathcal{D}$ and each morphism $f : A \to B$ in $\mathcal{C}$ with a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$. We require that functors preserve compositions of morphisms and identity morphisms (i.e., $F(f \circ g) = F(g) \circ F(f)$ for all morphisms $f, g$ in $\mathcal{C}$ and $F(\text{id}_A) = \text{id}_{F(A)}$ for all objects $A$ in $\mathcal{C}$).

**Definition 1.3.** Let $A_0$ and $A_1$ be topological vector spaces. Then $A_0$ and $A_1$ are said to be compatible if they are continuously embedded in some common Hausdorff topological vector space.

**Example 1.4.** If $p_0, p_1 \in [1, \infty]$, then $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$ are compatible, being continuously embedded in $L^0(\mathbb{R}^n)$ (the vector space of measurable functions on $\mathbb{R}^n$) equipped with the topology of convergence in measure (see Fremlin [3, Section 245]).

Recall that this topology is the one induced by the family of pseudometrics $\{\rho_E : |E| < \infty\}$, where $\rho_E(f, g) := \int_E \min\{|f-g|, 1\} \, dx$. It is Hausdorff because the Lebesgue measure on $\mathbb{R}^n$ is semifinite.
If $A_0$ and $A_1$ are compatible normed vector spaces, it can be shown that $A_0 \cap A_1$ and $A_0 + A_1$ are normed vector spaces with
\[
\|x\|_{A_0 \cap A_1} := \max \{\|x\|_{A_0}, \|x\|_{A_1}\} \quad \text{and} \quad \|x\|_{A_0 + A_1} := \inf_{x = x_0 + x_1, x_0 \in A_0, x_1 \in A_1} \|x_0\|_{A_0} + \|x_1\|_{A_1}.
\]

Moreover, if $A_0$ and $A_1$ are complete (i.e., are Banach spaces), then so are $A_0 \cap A_1$ and $A_0 + A_1$. The inclusions $A_0 \cap A_1 \subseteq A_0, A_1 \subseteq A_0 + A_1$ are also continuous. (See Bergh [1] Lemma 2.3.1 for proofs of these assertions.)

Henceforth, we will restrict our attention to the category $\mathcal{N}$ of normed vector spaces, whose morphisms are the bounded linear operators. In addition, whenever we consider a subcategory $\mathcal{C}$ of $\mathcal{N}$, we will assume that it is full.

**Definition 1.5.** Let $\mathcal{C}$ be a subcategory of $\mathcal{N}$. The category $\mathcal{C}_1$ of compatible couples in $\mathcal{C}$ is the category whose objects are all pairs $\vec{A} = (A_0, A_1)$ such that $A_0$ and $A_1$ are compatible spaces in $\mathcal{C}$ whose intersection and sum are also in $\mathcal{C}$, and whose morphisms $T : (A_0, A_1) \to (B_0, B_1)$ are all (bounded) linear operators from $A_0 + A_1$ to $B_0 + B_1$ whose restrictions $T|_{A_0} : A_0 \to B_0$ and $T|_{A_1} : A_1 \to B_1$ are bounded linear operators in $\mathcal{C}$.

**Definition 1.6.** Let $\mathcal{C}$ be a subcategory of $\mathcal{N}$ and $\vec{A} = (A_0, A_1)$ be a couple in $\mathcal{C}_1$. An intermediate space $A$ between $A_0$ and $A_1$ (or with respect to $\vec{A}$) is a space in $\mathcal{C}$ such that $A_0 \cap A_1 \subseteq A \subseteq A_0 + A_1$ with continuous inclusions.

**Example 1.7.** If $p_0, p_1 \in [1, \infty]$ and $p$ is between $p_0$ and $p_1$, then $L^p(\mathbb{R}^n)$ is an intermediate space between $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$ (taking, for instance, $\mathcal{C}$ to be all of $\mathcal{N}$).

**Definition 1.8.** Let $\vec{A}$ and $\vec{B}$ be couples in $\mathcal{C}_1$. Two spaces $A$ and $B$ in $\mathcal{C}$ are said to be interpolation spaces with respect to $\vec{A}$ and $\vec{B}$ if $A$ and $B$ are intermediate spaces with respect to $\vec{A}$ and $\vec{B}$, respectively, and if $T : A \to B$ is a bounded linear operator whenever $T : A \to B$ is.

We shall be particularly interested in the case where we can say the following about the norm of $T$ as an operator from $A$ to $B$.

**Definition 1.9.** If the interpolation spaces $A$ and $B$ satisfy $\|T\|_{A \to B} \lesssim \|T\|_{A_0 \to B_0}^{1-\theta} \|T\|_{A_1 \to B_1}^\theta$ for some $\theta \in (0, 1)$, then they are said to be of exponent $\theta$. If this bound holds with $\lesssim$ in place of $\lesssim$, then they are said to be exact of exponent $\theta$.

**Example 1.10.** The Riesz-Thorin theorem states that $L^{p_0}(X)$ and $L^{p_1}(Y)$ are exact interpolation spaces of exponent $\theta$ with respect to $(L^{p_0}(X), L^{p_1}(X))$ and $(L^{p_0}(Y), L^{p_1}(Y))$.

**Interpolation functors**

Having established the notion of interpolation spaces, we now turn to their construction.

**Definition 1.11.** Let $\mathcal{C}$ be a subcategory of $\mathcal{N}$. An interpolation functor is a functor $F$ from $\mathcal{C}_1$ to $\mathcal{C}$ such that if $\vec{A}$ and $\vec{B}$ are couples in $\mathcal{C}_1$, then $F(\vec{A})$ and $F(\vec{B})$ are interpolation spaces with respect to $\vec{A}$ and $\vec{B}$. In addition, we must have $F(T) = T$ for all morphisms $T$ in $\mathcal{C}_1$.

If, moreover, $F(\vec{A})$ and $F(\vec{B})$ are (exact) of exponent $\theta$ for all $\vec{A}$ and $\vec{B}$, we say that $F$ is (exact) of exponent $\theta$. 
Example 1.12. The functors $\Delta(\bar{A}) := A_0 \cap A_1$ and $\Sigma(\bar{A}) := A_0 + A_1$ are interpolation functors.

The interpolation functor we will use to derive decoupling estimates is denoted $C_\theta$ and is a functor from $B_1$ to $B$, where $B$ is the category of Banach spaces and $\theta \in (0, 1)$. The space $C_\theta(\bar{A})$ is commonly abbreviated as $\bar{A}[\theta]$. We will treat this functor as a ‘black box”; its definition can be found in Bergh [1, Section 4.1]. Its significance is owed to the following properties.

Theorem 1.13. The functor $C_\theta$ is exact of exponent $\theta$ for all $\theta \in (0, 1)$.

Proof. See Bergh [1, Theorem 4.1.2].

Theorem 1.14. Let $X$ be a $\sigma$-finite measure space and $p_0, p_1 \in [1, \infty]$. Then

$$(L^{p_0}(X), L^{p_1}(X))[\theta] = L^{p_\theta}(X)$$

for all $\theta \in (0, 1)$.

Proof. See Bergh [1, Theorem 5.1.1].

Definition 1.15. Let $A$ be a Banach space. For $p \in [1, \infty]$, the space $\ell^p(A)$ is the normed vector space of sequences $\{a_j\}_{j=1}^{\infty} \subseteq A$ such that

$$\|\{a_j\}\|_{\ell^p(A)} := \left(\sum_{j=1}^{\infty} \|a_j\|^p_A\right)^{1/p} < \infty,$$

with the obvious modification if $p = \infty$.

Remark. It can be shown that $\ell^p(A)$ is itself a Banach space. In fact, such spaces are special cases of Bochner spaces $L^p(X; A)$ – which are Banach spaces for all $p$ – where the sum above is replaced by an integral over some measure space $X$ (with some measurability restrictions on the functions being integrated).

Theorem 1.16. Let $\bar{A}$ be a compatible couple of Banach spaces and $p_0, p_1 \in [1, \infty]$. Then

$$(\ell^{p_0}(A_0), \ell^{p_1}(A_1))[\theta] = \ell^{p_\theta}((A_0, A_1)[\theta])$$

for all $\theta \in (0, 1)$.

Proof. See Bergh [1, Theorem 5.1.2].

(The proof is essentially an extension of the usual proof of the Riesz-Thorin theorem to accommodate analytic Banach space-valued functions defined on the strip $\{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$.)

Applications of interpolation to decoupling estimates

Finally, we recognize that the expressions on the right-hand sides of (1) and (2) are $\ell^p(L^p)$ and $\ell^2(L^p)$ norms, respectively, with sequences replaced by $N$-tuples in Definition 1.15. The estimates can therefore be written equivalently as

$$\|T\|_{\ell^p(L^p) \to L^p} \leq A(N) \quad (p > 2)$$
and
\[ \|T\|_{\ell^2(L^p) \to L^p} \leq B(N), \]  
where \( T\vec{f} := T((f_1, \ldots, f_N)) := \sum_{j=1}^N f_j \). In the following examples, “interpolating” refers to the application of the theorems in the preceding section.

Example 1.17. By the triangle inequality, \( \|T\|_{\ell^1(L^p) \to L^p} \leq 1 \). Interpolating with respect to \((\ell^p(L^p), \ell^1(L^p))\) and \((L^p, L^p)\) given (1) yields (2) with \( B(N) = A(N)^{1-\theta}1^\theta = A(N)^{p'/2} \) (which is a better constant than \( A(N) \)).

Example 1.18. By (2) with \( p = 1 \) and \( q = \infty \), we have \( \|T\|_{\ell^\infty(L^p) \to L^p} \leq N \|T\|_{\ell^\infty(L^p)} \), which implies that \( \|T\|_{\ell^\infty(L^p) \to L^p} \leq N \). Interpolating with respect to \((\ell^\infty(L^p), \ell^2(L^p))\) and \((L^p, L^p)\) given (2) yields (1) with \( A(N) = N^{1-\theta}B(N)^\theta = N^{1-2/p}B(N)^{2/p} \), which is a worse constant than \( B(N) \) if \( B(N) \lesssim N \). But (2) holds in general with \( B(N) = N^{1/2} \), so we incur a loss when converting from (2) to (1).

Alternatively, we can apply (2) directly to \( \vec{x} = (\|f_1\|_p, \ldots, \|f_N\|_p) \), which immediately gives \( A(N) = N^{1/2-1/p}B(N) \). This is always worse than \( B(N) \), but is actually better than \( N^{1-2/p}B(N)^{2/p} \) for \( B(N) \lesssim N^{1/2} \).

Example 1.19. By the triangle inequality, \( \|T\|_{\ell^2(L^\infty) \to L^\infty} \leq \sqrt{N} \). If the \( f_j \) are pairwise orthogonal in \( L^2 \), we also have \( \|T\|_{\ell^2(L^2) \to L^2} \leq 1 \) by the Pythagorean theorem. Interpolating with respect to \((\ell^2(L^\infty), \ell^2(L^2))\) and \((L^\infty, L^2)\) yields \( \|T\|_{\ell^2(L^p) \to L^p} \leq (N^{1/2})^{1-\theta}1^\theta = N^{1/2-1/p} \) for all \( p \in [2, \infty] \).

References